Generation of longitudinal vibrations in piano strings: From physics to sound synthesis[a)]

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Longitudinal vibration of piano strings greatly contributes to the distinctive character of low piano notes. In this paper a simplified modal model is developed, which describes the generation of phantom partials and longitudinal free modes jointly. The model is based on the simplification that the coupling from the transverse vibration to the longitudinal polarization is unidirectional. The modal formulation makes it possible to predict the prominent components of longitudinal vibration as a function of transverse modal frequencies. This provides a qualitative insight into the generation of longitudinal vibration, while the model is still capable of explaining the empirical results of earlier works. The semi-quantitative agreement with measurement results implies that the main source of phantom partials is the transverse to longitudinal coupling, while the string termination and the longitudinal to transverse coupling have only small influence. The results suggest that the longitudinal component of the tone can be treated as a quasi-harmonic spectrum with formantlike peaks at the longitudinal modal frequencies. The model is further simplified and applied for the real-time synthesis of piano sound with convincing sonic results. [DOI: 10.1121/1.1868212]

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I. INTRODUCTION

In this paper the generation mechanism of longitudinal vibration in piano strings is investigated. The purpose of this paper is twofold: to explain the experimental results of earlier papers and to provide a guideline for physics-based sound synthesis.

The importance of longitudinal vibration of piano strings was recognized long ago by piano builders. Conklin[1] demonstrated that the pitch relation of the transverse and longitudinal component strongly influences the quality of the tone and described a method to tune these components. Giordano and Korty[2] found that the amplitude of the longitudinal vibration is a nonlinear function of the amplitude of transverse vibration, confirming the assumption that the longitudinal component is generated by the nonlinearity of the string and not by the ‘‘misalignment’’ of the hammer.

Nakamura and Naganuma[3] found a second series of partials in piano sound spectra having one-fourth of inharmonicity compared to the main partial series. They attributed these to the horizontal polarization of the string, but they have actually found the series that later was named ‘‘phantom partials’’ by Conklin. Conklin[4] pointed out that the phantom partials are generated by nonlinear mixing and their frequencies are the sum or difference of transverse model frequencies. He named ‘‘even phantoms’’ those having double the frequency $(2f_n)$, of a transverse mode and ‘‘odd phantoms’’ those which appear at the sum f_m+f_n or difference f_m-f_n frequencies of two transverse modes. Conklin's measurements have shown that odd phantoms generally originate from adjacent parents, i.e., can be found at f_5+f_6 rather than at f_4+f_7. Phantom partials have also been found in the spectrum of guitar tones.[4] In a recent paper about guitar transients, Woodhouse states that the amplitude of the phantom partials seems to be modulated according to the longitudinal modal frequencies.[5] The present paper gives a theoretical explanation for these experimental results.

In an earlier work[6] some of the properties of phantom partials and longitudinal modes have been investigated. It was pointed out and it is emphasized here again that longitudinal modes and phantom partials are two different manifestations of the same phenomenon: they are the free and forced response of the same system, respectively. Therefore, in the theoretical treatment of the present paper they are covered jointly. This paper outlines some of the findings of Ref. 6 and provides a more refined theoretical background.

Theoretical works on nonlinear string vibrations that consider longitudinal motion include the papers of Narasimha,[7] Anand,[8] Watzky,[9] and O'Reilly and Holmes.[10] As these papers discuss the nonlinear coupling of the first modes of the two transverse polarizations, the inertial effects of the longitudinal vibration are neglected. This leads to the uniformity of the tension and equals to computing the tension from the relative elongation of the string.[8] (This is also studied in the Appendix.) The uniform tension approximation (thoroughly discussed by Legge and Fletcher[11]) often forms the basis for the sound synthesis of nonlinear strings. Papers include tension-modulated string models[12,13] based on digital waveguides[14] and energy-conserving finite difference schemes.[15] The inertial effects of longitudinal modes have

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been considered in the case of rubberlike strings by Leamy and Gottlieb[16] and by Kurmyshev.[17]

However, the results of these earlier papers[7–11,16,17] cannot be directly applied for the present purposes. This is because the present problem is more complex in the sense that not only the first few but 50 to 100 transverse modes have to be taken into account in the case of a struck piano string. On the other hand, it is simpler in the way that the longitudinal to transverse coupling and the coupling of different transverse modes are not investigated, as the primary interest is on the longitudinal vibration itself. Therefore, in this paper a modal model is developed that computes the spectrum of the longitudinal vibration in the case of arbitrary transverse model frequencies. It turns out that the most important properties of the longitudinal vibration can be explained by this simplified modal approach. The Appendix shows that the uniform tension approximation of earlier papers[7–11] is the special case of the tension computed from the modal model presented here.

The synthesis of the longitudinal components in piano tone is quite a recent topic. In an earlier work[6] the digital waveguide string model[14] has been extended by an auxiliary digital waveguide for computing the response of the phantom partials and by second-order resonators for modeling the longitudinal free modes. A very efficient method for modeling the phantom partials was proposed by Bensa.[18] The model had a loose connection to physical reality, since in that work spatially uniform tension was assumed. In a recent work[19] a physics-based solution is presented for modeling the longitudinal components jointly (i.e., phantom partials and free modes together), which is able to produce high quality piano sounds. In this model second-order resonators are nonlinearly excited according to the transverse string shape computed by a finite difference model. This paper outlines the above approach and presents an alternative technique having lower computational cost.

The paper is organized as follows: first the differential equations are derived from basic principles in Sec. II. In Sec. III a modal model is presented, which analytically computes the longitudinal vibration under certain assumptions. In Sec. IV the measurements of other authors[2–4] are explained by the results of the proposed model, and some measurement results of the present authors are also given. Section V describes efficient algorithms for the synthesis of piano sound including the longitudinal vibrations. The Appendix relates the uniform tension approximation to the modal model presented in this paper.

II. EQUATIONS FOR ONE PLANE OF VIBRATION

A real piano string is vibrating in two transverse planes, and in the longitudinal direction as well. Principally, piano hammers excite one transverse polarization of the string, while the other two polarizations are gaining energy through coupling. For simplicity, it is assumed in this section that the string is vibrating in one plane, i.e., one transverse and one longitudinal polarization are present. Losses and dispersion are neglected for the time being and they are included later in Sec. III. The differences arising from the incorporation of two transverse polarizations are outlined in Sec. III E. Note that similar derivations in a different formulation can be found in earlier works, e.g., in the textbook of Morse and Ingard.[20] The equations are developed here to make the paper self-explanatory.

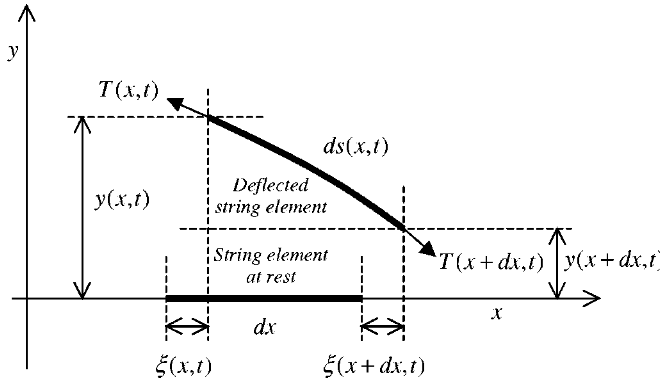

FIG. 1. The string element.

When a transverse displacement occurs on the string, the string elongates. This results in a force exciting a longitudinal wave in the string. The longitudinal wave modulates the tension of the string, which influences the transverse vibration.

The element of length dx at equilibrium will have the length ds, as depicted in Fig. 1, which is calculated as follows:

$$ds^2(x,t) \approx [\xi(x+dx,t) - \xi(x,t) + dx]^2 + [y(x+dx,t) - y(x,t)]^2. \tag{1}$$

As dx is infinitesimally small, the differences are substituted by differentials

$$ds = \sqrt{\left(\frac{\partial \xi}{\partial x} + 1\right)^2 dx^2 + \left(\frac{\partial y}{\partial x}\right)^2 dx^2}, \tag{2}$$

where $y = y(x,t)$ and $\xi = \xi(x,t)$ are the transverse and longitudinal displacements of the string with respect to time t and space x. The tension $T = T(x,t)$ of the string (which equals to T_0 at rest) is calculated according to Hooke's law,

$$T = T_0 + ES\left(\frac{ds}{dx} - 1\right), \tag{3}$$

where E is the Young's modulus and S is the cross-section area of the string. By substituting Eq. (2) into Eq. (3) and neglecting the higher order terms the tension can be approximated as

$$T \approx T_0 + ES\left[\frac{\partial \xi}{\partial x} + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]. \tag{4}$$

As the segment ds is nearly parallel to the x axis, the longitudinal force on the segment ds can be approximated as the difference of the tension at the sides of the segment

$$F_x \approx \frac{\partial T}{\partial x} dx \approx ES\left[\frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2}\frac{\partial(\partial y/\partial x)^2}{\partial x}\right] dx. \tag{5}$$

Note that the resolution of the force would only introduce a negligible correction term for metal strings (where $T_0 \ll ES$). The force F_x acts on a mass μdx where μ is the mass per unit length. Accordingly, the longitudinal vibration is approximately described by

$$\mu \frac{\partial^2 \xi}{\partial t^2} = ES \frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2} ES \frac{\partial(\partial y/\partial x)^2}{\partial x}, \tag{6}$$

which is the standard one-dimensional wave equation with an additional force term depending on the transverse vibration. According to Eq. (6), the transverse string motion can only excite the longitudinal vibration if the square of the string slope is significant, i.e., the transverse displacement is relatively large. Note that neglecting the term $\mu(\partial^2 \xi/\partial t^2)$ leads to the uniform tension approximation (see the Appendix).

After similar derivations the wave equation for the transverse motion can be written as

$$\mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2} + ES \frac{\partial\{(\partial y/\partial x)[\partial \xi/\partial x + \frac{1}{2}(\partial y/\partial x)^2]\}}{\partial x}, \tag{7}$$

which is again a one-dimensional wave equation with an additional force term depending on the product of the transverse slope and the tension variation. Consequently, the longitudinal vibration influences the transverse one if both the transverse and longitudinal displacements are relatively large.

To sum up, both Eqs. (6) and (7) can be considered as standard linear wave equations with additional forcing terms on their right-hand sides. From Eq. (6) it can be concluded that the level of transverse to longitudinal coupling depends on the magnitude of transverse vibration according to a square law. From Eq. (7) it turns out that the amount of longitudinal to transverse coupling is a third-order function of the amplitude of transverse vibration, since ξ is in the order of y^2 [see Eq. (6)].

III. TRANSVERSE TO LONGITUDINAL COUPLING

As the main interest of this paper is to clarify the generation of longitudinal string vibration, a further simplification is made: the longitudinal to transverse coupling is neglected. Moreover, the string termination is assumed to be infinitely rigid. The effects of these assumptions are covered in Sec. III F.

These limitations lead to a model that cannot be in complete quantitative agreement with measurements. On the other hand, its simplicity helps to gain a better understanding of the phenomenon. This simple model is already enough for the qualitative explanation of the measurements of earlier papers,[2–4] as discussed in Sec. IV later. Furthermore, sound synthesis models (see Sec. V) based on these principles produce realistic piano sounds.

A. Excitation force

For the freely vibrating, dispersive, lossy, and rigidly terminated string the transverse displacement for a given position $0 \leqslant x \leqslant L$ and time $t \geqslant 0$ can be written in the following form:[21]

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos(2\pi f_n t + \varphi_n) e^{-t/\tau_n} \sin\left(\frac{n\pi x}{L}\right), \tag{8}$$

where f_n is the frequency, τ_n is the decay time, A_n is the initial amplitude, and φ_n is the initial phase of the transverse mode n, and L refers to the length of string. This form is of particular interest since the motion of the piano string behaves similarly after the hammer-string contact (typically after 1–2 ms).

As a general case, the transverse displacement of a rigidly terminated string can be described by a similar formula:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin\left(\frac{n\pi x}{L}\right), \tag{9}$$

where the time-dependent terms of Eq. (8) are substituted by the series of functions $y_n(t)$, which can be considered as the instantaneous amplitudes of the modes $n=1,...,\infty$. This notation is used to obtain simpler formulas in the following derivations.

According to Eq. (6), the transverse to longitudinal excitation-force distribution $F_{t\to l}(x,t)$ is computed as

$$F_{t\to l}(x,t) = \frac{1}{2} ES \frac{\partial[\partial y(x,t)/\partial x]^2}{\partial x}. \tag{10}$$

Substitution of Eq. (9) into Eq. (10) yields

$$F_{t\to l}(x,t) = \frac{1}{2} ES \frac{\partial[\sum_{n=1}^{\infty} y_n(t)(n\pi/L)\cos(n\pi x/L)]^2}{\partial x}, \tag{11}$$

which, after some derivations, takes the following form:

$$F_{t\to 1}(x,t) = -ES \frac{\pi^3}{4L^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} y_m(t) y_n(t) mn \times \left[(m+n) \sin\left(\frac{m+n}{L}\pi x\right) + (m-n) \sin\left(\frac{m-n}{L}\pi x\right) \right]. \tag{12}$$

Note that the indices m and n belong to variables of transverse modes throughout the paper. The variables of longitudinal modes are indexed by k.

B. Longitudinal motion

The longitudinal displacement can be written in the same form as Eq. (9), resulting in

$$\xi(x,t) = \sum_{k=1}^{\infty} \xi_k(t) \sin\left(\frac{n\pi x}{L}\right). \tag{13}$$

By applying the derivations of Morse[22] (which were originally developed for transverse vibrations), the instantaneous amplitude $\xi_k(t)$ of the longitudinal mode k is obtained as

$$\xi_k(t) = F_{t\to l,k}(t) * \xi_{\delta,k}(t), \tag{14a}$$

$$F_{t\to l,k}(t) = \int_0^L F_{t\to 1}(x,t) \sin\left(\frac{k\pi x}{L}\right) dx, \tag{14b}$$

$$\xi_{\delta,k}(t) = \frac{1}{\pi L \mu} \frac{e^{-t/\tau_k'}}{f_k'} \sin(2\pi f_k' t), \tag{14c}$$

where the $*$ sign denotes time-domain convolution and $F_{t\rightarrow 1,k}(t)$ is the excitation force acting on the longitudinal mode k. The time-domain impulse response of longitudinal mode k is denoted by $\xi_{\delta,k}(t)$, where f'_k and τ'_k stand for the frequency and decay time of the longitudinal mode k. Note that the single quote in f'_k and τ'_k is used to distinguish the longitudinal variables from the transverse ones.

The first step in calculating the longitudinal motion is the computation of the excitation force $F_{t\rightarrow l,k}(t)$ by Eq. (14b), which is the scalar product of the excitation-force distribution $F_{t\rightarrow l}(x,t)$ and the longitudinal modal shape. From Eqs. (12) and (14b) it follows that $F_{t\rightarrow l,k}(t)$ is nonzero for $m+n=k$ and $|m-n|=k$ only, since in all other cases the spatial distribution of the excitation $F_{t\rightarrow l}(x,t)$ is orthogonal to the modal shape of mode k.

The two cases can be computed separately by defining $F_{t\rightarrow l,k}(t)$ as a sum of two components, i.e., $F_{t\rightarrow l,k}(t) = F_{t\rightarrow l,k}(t)^{+} + F_{t\rightarrow l,k}(t)^{-}$. The component originating from the transverse modes that satisfy $m+n=k$ is

$$F_{t\rightarrow 1,k}(t)^{+} = -ES\frac{\pi^3}{8L^2}\sum_{n=1}^{k-1} y_{k-n}(t)y_n(t)k(k-n)n. \tag{15a}$$

The component coming from $|m-n|=k$ becomes

$$F_{t\rightarrow 1,k}(t)^{-} = -2ES\frac{\pi^3}{8L^2}\sum_{n=1}^{\infty} y_{k+n}(t)y_n(t)k(k+n)n. \tag{15b}$$

The factor of 2 in Eq. (15b) comes from the fact that there are two equal series $m=k+n$ and $n=k+m$, since both satisfy $|m-n|=k$.

C. Excitation frequencies

For qualitative understanding of the longitudinal components it is useful to look at the spectra of the excitation force series $F_{t\rightarrow l,k}(t)$. The most important question is where the frequency peaks can be found.

To the first approximation, the instantaneous amplitudes $y_n(t)$ are exponentially decaying sinusoidal functions with the frequencies f_n, such as in Eq. (8). By observing Eqs. (15a) and (15b) the frequencies of the mixing terms in $F_{t\rightarrow l,k}(t)$ can be calculated as

$$\begin{aligned}
&\text{Frequencies}\\
&\text{in } F_{t\rightarrow l,k}(t)^{+}: \begin{cases} f_n + f_{k-n} \approx f_k, \\ f_n - f_{k-n} \approx f_{|2n-k|}, \end{cases}\\
&\text{in } F_{t\rightarrow l,k}(t)^{-}: \begin{cases} f_n + f_{k+n} \approx f_{2n+k}, \\ f_n - f_{k+n} \approx f_k, \end{cases}
\end{aligned} \tag{16}$$

where the form f_a refers to the frequency of the transverse mode with mode number a.

1. Harmonic transverse vibration

The approximations in Eq. (16) become equalities if the transverse frequencies f_n are perfectly harmonic, i.e., $f_n = nf_0$, which is the case for string instruments having negligible inharmonicity. In this case, there is a strong peak at the frequency f_k, and a series of partials at f_{2n-1} for odd k and at f_{2n-2} for even k, with $n=1,...,\infty$. This means that the odd longitudinal modes are excited by components having the same frequencies as the odd transverse modes. Similarly, the even longitudinal modes are excited at the even transverse modal frequencies.

These frequencies form the inputs of the time-domain impulse responses $\xi_{\delta,k}(t)$, which can be considered as second-order resonators [see Eq. (14c)]. The output of a resonator has two types of components: one component is the free response, which is a decaying sinusoid at the frequency f'_k. The other component is the forced response consisting of the frequency series f_{2n-1} or f_{2n-2} with $n=1,...,\infty$. The amplitudes of these spectral lines are amplified around the peak of the resonator f'_k. As the responses of all the longitudinal modes are summed together, the output becomes similar to having formants on a rich harmonic spectrum. In any case, the forced components are indistinguishable from the transverse ones since they are exactly at the same frequencies.

2. Inharmonic transverse vibration

In the case of the piano, due to the stiffness of the string, the transverse partial frequencies do not form a perfect harmonic series but rather obey the equation[23]

$$f_n = f_0 n\sqrt{1+Bn^2}, \tag{17}$$

where B is the inharmonicity coefficient and $f_0 \approx f_1$ is the fundamental frequency of the string.

In this case the terms $f_n + f_{k-n}$ and $f_n - f_{k+n}$ do not have the frequency of f_k but form a bunch of peaks around f_k. The peaks at the frequencies $f_n - f_{k-n}$ lie somewhat higher compared to $f_{|2n-k|}$ and the frequencies $f_n + f_{k+n}$ are lower than f_{2n+k}. This means that these peaks depart from the transverse partials in a rate determined by the inharmonicity coefficient B and the longitudinal mode number k. However, it is still true that odd longitudinal modes are excited by an oddlike partial series, while even longitudinal modes are excited by an evenlike one.

The force exciting the first longitudinal mode $F_{t\rightarrow 1,1}(t)$ is displayed in Fig. 2(a) by a solid line, computed by the resonator-based string model (see Sec. V C), which is the discrete-time implementation of the modal model described in Secs. III A and III B. Note that the excitation force has an oddlike partial series. The spectrum of the transverse bridge force is displayed by dots to show the transverse modal frequencies as a reference. The dashed line indicates the Fourier transform of the impulse response of the first longitudinal mode $\xi_{\delta,1}(t)$, amplifying the frequencies around 690 Hz. Figure 2(b) shows the excitation-force spectrum of the second longitudinal mode for the same example. It can be seen that here the excitation spectrum contains even partials only and that the peak of the longitudinal mode (dashed line) is located at a higher frequency (1380 Hz in this case).

The longitudinal motion is the sum of the motion of different modes. This means that spectra similar to Figs. 2(a) and (b) should be superimposed with slightly shifted excitation frequencies and very different longitudinal modal frequencies. The result is similar to formants on a quasiharmonic spectrum but here the peaks are somewhat smeared as they are made up of many close frequencies. The most

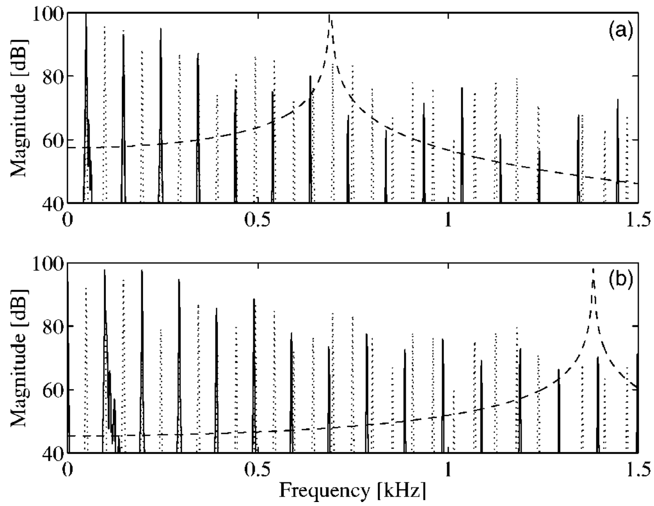

FIG. 2. The force spectrum exciting the first (a) and the second (b) longitudinal modes [$F_{t\rightarrow l,1}(t)$ and $F_{t\rightarrow l,2}(t)$] computed by the resonator-based string model of Sec. V C (displayed by solid line). The transverse bridge force (dotted line) is displayed to show the transverse modal frequencies. The dashed line shows the frequency response of the first (a) and the second (b) longitudinal modes. The relative levels of the signals are arbitrary.

important difference from the case of harmonic transverse vibration is that these smeared peaks appear between the transverse ones and therefore they can be easily distinguished.

D. Longitudinal force at the bridge

The longitudinal component of the tone is transmitted to the soundboard via the force acting on the bridge in the longitudinal direction. This force equals to the tension [see Eq. (4)] at the termination $x=L$, written as

$$F_l(t) = -\left[T_0 + ES\frac{\partial \xi}{\partial x}\bigg|_{x=L} + \frac{1}{2}ES\left(\frac{\partial y}{\partial x}\bigg|_{x=L}\right)^2 \right], \quad (18)$$

showing that the force $F_l(t)$ depends not only on the longitudinal motion but on the transverse vibration as well.

The force component coming from the transverse motion have the same sum- and difference-frequency terms as the component arising from the longitudinal motion, but their amplitudes are different. It is an interesting outcome that when the transverse motion of the string contains low frequency components only, most of these terms cancel out and only the double frequency terms remain. This produces the same longitudinal bridge force as what would occur with applying the uniform tension approximation.[11] The derivation of this result is included in the Appendix.

E. Extension to two transverse planes

Real strings vibrate in two transverse polarizations. The modal frequencies for these polarizations can be different for the same modes, mostly because of the direction-dependent termination impedance. This produces beating and two-stage decay in piano sound.[24] Working out the equations (1)–(6) for the three-dimensional case gives

$$\mu\frac{\partial^2 \xi}{\partial t^2} = ES\frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2}ES\left[\frac{\partial(\partial y/\partial x)^2}{\partial x} + \frac{\partial(\partial z/\partial x)^2}{\partial x}\right], \quad (19)$$

where z is the string displacement in the direction perpendicular the already considered transverse y and longitudinal x directions.

It follows from Eq. (19) that the excitation-force distribution $F_{t\rightarrow l}(x,t)$ is the superposition of the excitation-force distributions computed for the two transverse planes separately. Accordingly, if two modes vibrate in two planes perpendicular to each other, their sum and difference frequencies do not appear in the excitation force. In reality the vibrating planes are not perfectly perpendicular to each other (the motion is actually not even planar[7–10]), meaning that mode m vibrating in one plane will mix with mode n vibrating in a different plane.

The modal frequencies $f_{n,1}$ and $f_{n,2}$ of the two transverse polarizations are slightly different for the same mode number n. Accordingly, the excitation components coming from the transverse modes m and n consist of four different frequencies. For example, the sum-frequency components have the frequencies $f_{m,1}+f_{n,1}$, $f_{m,1}+f_{n,2}$, $f_{m,2}+f_{n,1}$, and $f_{m,2}+f_{n,2}$. The difference-frequency components can be expressed similarly.

F. Validity of the approximation

1. Neglecting the longitudinal to transverse coupling

The assumption of neglecting the longitudinal to transverse coupling is valid until the longitudinal vibration is small compared to the transverse one. However, if one of the excitation frequencies of the longitudinal mode k [see Eq. (16)] is very close to the resonant frequency f'_k of that mode, the longitudinal motion can have extremely large amplitude. This would not happen in reality since the longitudinal motion would diminish the amplitude of those transverse modes from which it originates (the total energy of transverse and longitudinal vibrations cannot increase). This stabilizing effect is not included in our modal model. On the other hand, these coincidences have a small practical significance from the sound synthesis viewpoint since they produce an unpleasant ringing sound even when computed by a finite difference string model (see Sec. V A) having bidirectional coupling. (This fact also implies that these annoying coincidences should also be avoided in real pianos by careful string- and scale design.)

The longitudinal to transverse coupling would also introduce some terms of third order in the amplitude of the transverse vibration, but their contribution is less significant compared to the second-order terms discussed here. To sum up, the frequencies predicted by the model of Sec. III should be in quantitative agreement with the dominant peaks found in real piano spectrum. The amplitude behavior is described properly for those peaks that do not coincide with the resonant frequency of the excited longitudinal mode. (This holds for most of the peaks.)

2. Perfectly rigid termination

The termination of piano strings is not perfectly rigid, contrary to the assumptions made here. As the impedance of the bridge is ca. 1000 times larger compared to the impedance of the string, its main effect is a change in the partial frequencies f_n and decay time τ_n, which can be easily incorporated in Eq. (8). The modal shapes also change slightly [L is substituted by $L+\delta L$ in Eq. (8), see Ref. 11], meaning that none of the longitudinal modal shapes are completely orthogonal to the modal shapes of the excitation-force distribution. However, it is still true that the dominant force components are those computed by Eqs. (15a) and (15b). This is confirmed by finite difference simulations showing only a small change in the output when a more realistic termination model is applied.

The termination of piano strings can also contribute to the energy transfer between the transverse and longitudinal motion.[11] As such a coupling is linear, it does not introduce new terms by itself. The transverse frequencies can appear in the longitudinal motion, and, conversely, the longitudinal frequencies may turn up in the transverse vibration. However, those transverse and longitudinal components that have the same frequency cannot be distinguished in the sound pressure. The coupling through the bridge in combination with the transverse to longitudinal coupling along the string could produce new terms, but they are of fourth order in the amplitude of the transverse vibration.

IV. CONNECTIONS TO MEASUREMENTS

In this section the results of Sec. III are related to the measurements of other authors.[2–4] On the one hand, this confirms the modal model presented in Sec. III. On the other hand, it helps to understand the theoretical reasons underlying the findings of these experimental studies.[2–4]

A. Parentage of phantom partials

From the theoretical point of view, phantom partials are the forced motion of longitudinal vibrations. An interesting property of odd phantom partials discovered by Conklin[4] is that they originate from adjacent parents, i.e., they can be found at frequencies f_m+f_n where $m-n=1$.

By looking at $F_{\mathrm{t}\rightarrow l,k}(t)^-$ in Eq. (16) it turns out that the frequencies $f_n+f_m=f_n+f_{k+n}$ are quite close to each other for $m+n=p$ (they would actually coincide in the case of a perfectly harmonic transverse vibration having the frequency f_{2n+k}). The question is which f_m+f_n combination has the largest amplitude in the resulting sound. It follows from Eq. (16) that the different f_m+f_n components belonging to the same smeared peak (i.e., $m+n=p$) excite different longitudinal modes. Namely, the frequency f_m+f_n excites the longitudinal mode having the mode number $k=m-n$. Accordingly, that f_m+f_n component results in the largest longitudinal motion which excites the longitudinal mode having a modal frequency f'_k close to the frequency f_m+f_n. In other words, if the frequency of a phantom partial group is close to the frequency f'_k of the kth longitudinal mode, it mainly originates from parents having mode number difference of k.

The lower odd phantom partials, which were measured by Conklin,[4] most probably have frequencies to which the first longitudinal mode is the nearest. In this case the f_m+f_n terms satisfying $m-n=1$ dominate, which actually originate from adjacent parents f_n and f_{n+1}.

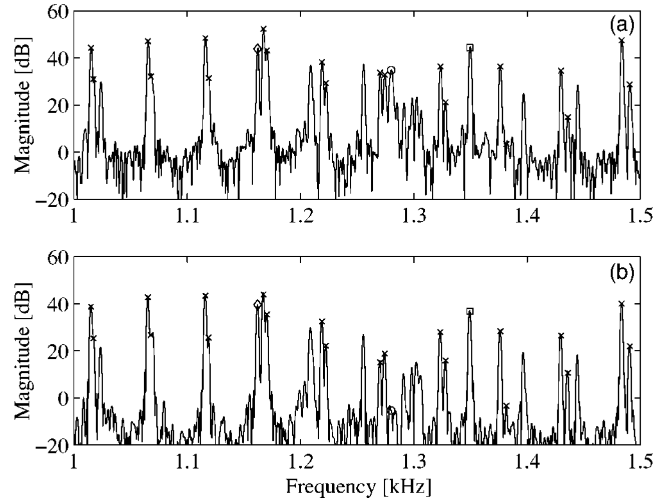

FIG. 3. Spectrum of the first (a) and the second second (b) of a F_1 piano tone. Transverse partials are marked by crosses and the second longitudinal mode is marked by a circle. Two prominent phantom partial groups are indicated by a square and a diamond (the latter is magnified in Fig. 4).

Similar considerations apply for even phantoms, that is, they are generated by parents having mode number difference of 2, 4, 6, etc., depending on the frequency of the phantom partial. However, there is an important difference that double frequency terms $2f_n$ also occur in the spectrum. These $2f_n$ components would arise even when the bandwidth of transverse components was significantly lower than the frequency of the first longitudinal mode, i.e., when the tension was approximately uniform along the string (see the Appendix). In other words, these are the only components that can be explained by the uniform tension approximation of Refs. 7–11, while for the sum- and difference frequency components the inertial effects of longitudinal modes have to be included in the model.

The spectrum of a recorded F_1 piano tone (having only one string) is displayed in Fig. 3. Transverse partials are indicated by crosses and the free response of the second longitudinal mode is marked by a circle. The remaining peaks are the forced response of the longitudinal motion, i.e., phantom partials. Figure 3(a) shows the first second of the tone and Fig. 3(b) displays the second, giving an insight to the evolution of the spectrum. Note that the free response of the longitudinal mode (circle) disappears fast in the noise (the decay time is ca. 0.15 s), while the phantom partials remain significant and their decay rate is comparable to that of the transverse partials. It can be said in general that the highest nontransverse peaks in the long-term spectrum are phantom partials amplified by a longitudinal mode (one prominent example is marked by a square). This suggests that the forced response of the longitudinal motion may have a larger perceptual significance than the free response itself. Most probably the pitch of the longitudinal component is determined by these amplified phantom partials (like the one marked by a square in Fig. 3) and not by the fast decaying free response. The interested reader may listen to the sound examples demonstrating the relative significance of these components.[25]

The "single" phantom partial marked by a diamond in

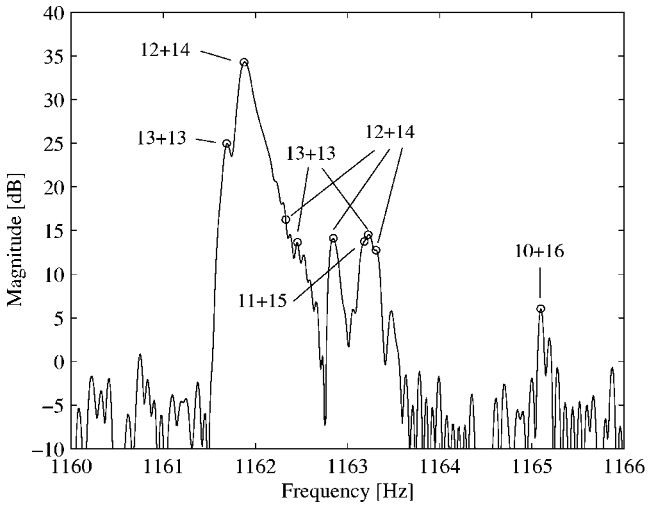

FIG. 4. The spectrum of an even phantom partial group in the F_1 piano tone of Fig. 3. Sum frequencies of transverse modes f_m+f_n are marked by circles, and the mode numbers of the parent partials are labeled in the form of $m+n$. The phantom group is displayed by a diamond in Fig. 3.

Fig. 3 becomes a group of partials when plotted at a higher frequency resolution in Fig. 4. In this case the data length is 16 s (705 600 samples at $f_s=44.1$ kHz), which was zero padded to 2^{22} samples after applying a Hanning window. The most prominent peaks of the phantom group are marked by circles. The label "$m+n$" beside a circle indicate that the circle is located at the sum frequency of the transverse modes m and n (i.e., at f_m+f_n). The frequencies of the transverse modes were determined by finding peaks in the spectrum. Note that the same $m+n$ combinations can be found at several peaks: the reason is that the two different frequencies $f_{m,1}$ and $f_{m,2}$ of the two transverse polarizations of mode m mix with the two different frequencies $f_{n,1}$ and $f_{n,2}$ of mode n, as predicted in Sec. III E.

It can be seen in Fig. 4 that the highest peak comes from the 12th and 14th transverse modes and not from the 13th mode itself, although the amplitude of the latter is only 10 dB smaller. Other even phantoms show the same phenomenon: they principally originate from parents having mode-number difference of 2, 4, etc., and not from a single mode by frequency doubling. This contradicts the findings of Conklin[4] but confirms the analysis of Sec. III.

B. Inharmonicity of phantom partials

Nakamura and Naganuma[3] found that the inharmonicity of phantom partials (called "lower series" in Ref. 3) is one-fourth of that of normal transverse partials.

This can be explained by knowing that phantom partials are mainly produced by parents with mode numbers close to each other. This means that even phantoms have an approximate frequency of $f_p=2f_n$, where $p=2n$ is the "mode number" of the phantom partial. (See Fig. 4 as an example, where $f_{12}+f_{14}\approx 2f_{13}$.) Writing $f_p=2f_n$ according to Eq. (17) and expressing the frequencies by the phantom mode number $p=2n$ gives[6]

$$f_p \approx 2f_n = 2f_0 n\sqrt{1+Bn^2} = f_0 p\sqrt{1+\tfrac{1}{4}Bp^2}. \qquad (20)$$

For even phantoms, the expression is quite accurate. For odd phantoms, $n=p/2$ is not an integer number. However, as the inharmonicity curve is a smooth function, the frequencies of odd phantom partials are also close to the ones predicted by Eq. (20).[6]

C. Amplitude of longitudinal vibration

Giordano and Korty[2] found that the amplitude of the longitudinal vibration is a nonlinear function of the amplitude of the transverse one. They noted that the nonlinear curve is faster than a simple quadratic function.

Equation (12) shows that a peak in the excitation spectrum of a longitudinal mode is a quadratic function of the overall amplitude of the generating transverse modes m and n. However, the amplitude of longitudinal motion is mostly determined by parents having sum frequencies f_m+f_n around the longitudinal modal frequencies f'_k. The amplitude of these parents (with mode numbers around 10–20 in practice) are a nonlinear function of the overall amplitude of the transverse vibration. This is because of the nonlinear nature of the hammer–string interaction (see, e.g., Ref. 26). The presence of this second kind of nonlinearity explains why Giordano and Korty[2] could not measure a second-order relationship.

V. SOUND SYNTHESIS

The original motivation of this research was to support the development of physics-based piano models. These physics-based models do not have to describe each part of the instrument precisely. They should be as simple as possible while still producing agreeable sound quality. Therefore these models are often constructed from precise physical descriptions by neglecting those effects that have small perceptual relevance.

A. Finite difference modeling

A straightforward approach of modeling the vibration of piano strings is implementing the simultaneous differential equations (6) and (7) by the finite difference approach. Naturally, these have to be extended by the terms realizing frequency-dependent losses and dispersion. In an earlier work[6] such a model was developed along the lines of the transverse string model of Chaigne and Askenfelt.[27,28] However, the computational demand of such an approach is large because high sampling frequency ($f_s\approx 500$ kHz) is required due to the higher propagation speed in the longitudinal direction. Still, this approach can be very useful for experimental purposes. A commercial computer program based on a finite difference string model was written by Bernhard,[29] helping piano tuners in scale design.

The inclusion of longitudinal components in the piano model greatly improves the quality of synthesized piano sounds. However, complete finite difference modeling of the string is too demanding for real-time sound synthesis applications. In the following sections two models are presented that overcome this limitation. The composite model of Sec. V B replaces the finite difference model implementing Eq. (6) with the modal description for the longitudinal polariza-

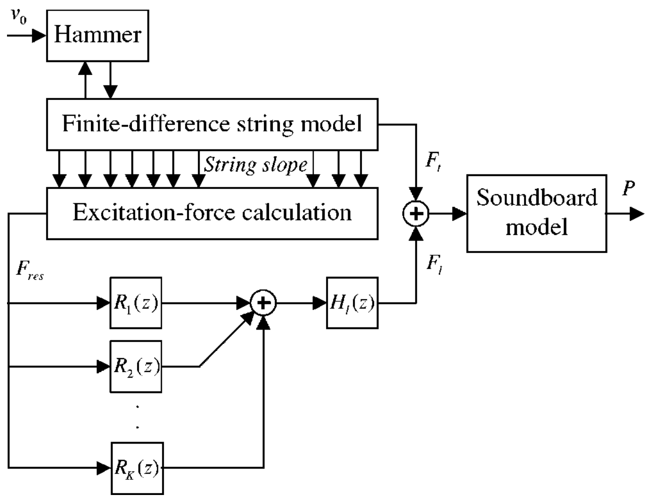

FIG. 5. The composite string model applying finite difference modeling and second-order resonators $R_1,\ldots,R_K$.

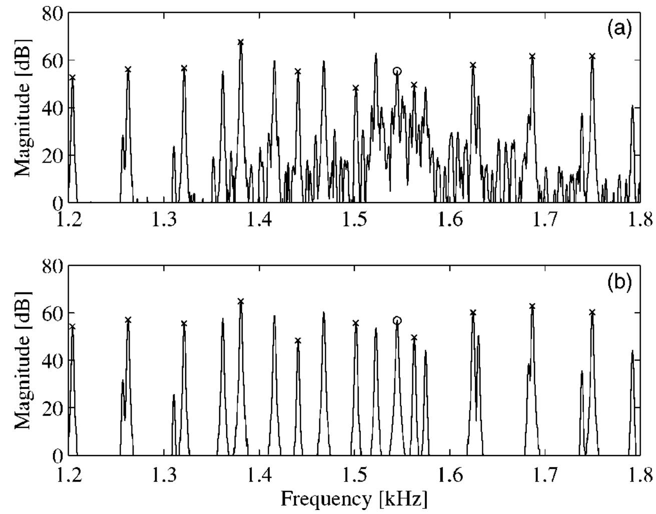

FIG. 6. The sound pressure spectrum of the first second of a synthesized G_1 piano tone[25] computed by the composite string model of Sec. V B (a) and by the resonator-based string model of Sec. V C (b). Crosses indicate the transverse partials and the second longitudinal mode is marked by a circle in both figures. To be compared with Fig. 3(a).

tion, while the transverse vibration is computed by a variation of Eq. (7). The resonator-based string model of Sec. V C applies the modal model for both polarizations instead of directly realizing Eqs. (6) and (7).

B. The composite string model

In an earlier work[19] a composite string model was introduced, which computes the longitudinal vibration in a modal form according to the theory presented in Sec. III. The model structure is depicted in Fig. 5. The transverse deflection $y(x,t)$ is computed by a finite difference string model running at audio sampling rate (e.g., $f_s = 44.1$ kHz), which implements the following differential equation:

$$\mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2} - ES\kappa^2 \frac{\partial^4 y}{\partial x^4} - 2b_1\mu \frac{\partial y}{\partial t} + 2b_2\mu \frac{\partial^3 y}{\partial x^2 \partial t}. \tag{21}$$

Equation (21) is a variation of Eq. (7) where the nonlinear forcing term [the rightmost term of Eq. (7)] is missing, as the longitudinal to transverse coupling is neglected, while it is extended by terms realizing dispersion and losses. This is basically the string model proposed by Chaigne and Askenfelt,[27,28] except for the last term, which substitutes temporal derivatives with spatial ones. This modification is suggested by Bensa *et al.*,[30] leading to a stable system for arbitrary b_2. The κ parameter in the dispersion term refers to the radius of gyration of the string, and the constants b_1 and b_2 determine the decay rates of the partials. A finite difference hammer model[27,28] is also attached to the string. The initial velocity of the hammer is denoted by v_0 in Fig. 5.

The excitation-force distribution of the longitudinal motion $F_{t\to l}(x,t)$ is computed according to Eq. (10) from this transverse displacement. Then the excitation force $F_{t\to l,k}(t)$ of each longitudinal mode k is computed by a scalar product with the longitudinal modal shape [see Eq. (14b)]. The instantaneous amplitudes $\xi_k(t)$ of the longitudinal modes are calculated according to Eq. (14a), which is implemented by second-order resonators ($R_1,\ldots,R_K$ in Fig. 5).

In order to reduce the computational cost, the same excitation force is used for all the longitudinal modes.[19] This is acceptable from a perceptual point of view because the spectra of the excitation forces $F_{t\to l,k}(t)$ are quite similar. It follows from Eq. (16) that odd phantoms arise from the vibration of odd longitudinal modes and even phantoms from the vibration of even ones. Therefore it is satisfactory to compute the excitation force of one odd and one even longitudinal mode, e.g., the input of the resonators $R_1,\ldots,R_K$ can be $F_{\mathrm{res}} = F_{t\to l,5}(t) + F_{t\to l,6}(t)$. This way a multiple (or smeared) peak of a phantom partial is substituted by a single, exponentially decaying sinusoid. In order to avoid an unpleasant ringing sound (see Sec. III F 1), the frequencies f'_k of the resonators $R_1,\ldots,R_K$ are set in a way that they do not coincide with the peaks of their excitation signal F_{res}. It has been found that using a common excitation force for $R_1,\ldots,R_K$ does not impair sound quality, but contributes to large computational savings.

The force signals F_t and F_l in Fig. 5 coming from the transverse and longitudinal polarizations are sent to the soundboard model, which computes the sound pressure P. The soundboard is modeled by a multi-rate filtering algorithm approximating the measured impulse response of a transversely excited piano soundboard.[19] The soundboard responds differently to a longitudinal force than to a transverse one. This difference is modeled by a simple high-pass filter $H_l(z)$ in the longitudinal force path.

The sound pressure spectrum of the first second of a synthesized G_1 note[25] is displayed in Fig. 6(a). The phantom partials are clearly visible between the transverse modes, which are emphasized around the longitudinal free mode at 1450 Hz. The circle indicates the component coming from the longitudinal free response, while the crosses show the transverse modal frequencies. It can be seen that the spectrum is similar to that of a real piano tone displayed in Fig. 3. The composite string model produces the same sound quality as the full finite difference method of Sec. V A, while its computational requirements are reduced by an order of a magnitude (to around 10%–15%).

C. The resonator-based string model

The resonator-based string model is the discretization of the equations presented in Sec. III. The string displacement is represented by its modal form [see Eqs. (9) and (13)] for both the transverse and longitudinal polarizations and the instantaneous amplitudes $y_n(t)$ and $\xi_k(t)$ are computed by second-order resonators.

The string is excited by a hammer in the transverse polarization. The hammer is modeled in the same way as in the case of finite difference string models.[27,28] The string response to the hammer force is calculated by a set of second-order resonators, which have input and output coefficients depending on the hammer position. The outputs of these resonators correspond to the instantaneous amplitudes $y_n(t)$ of the transverse vibration, which can be directly used to compute the excitation force $F_{t\to l,k}(t) = F_{t\to l,k}(t)^+ + F_{t\to 1,k}(t)^-$ of the longitudinal modes by using Eqs. (15a) and (15b). From this point, the approach is the same as taken in Sec. V B: the excitation force of one even and one odd longitudinal mode is calculated and summed [e.g., $F_{res} = F_{t\to l,5}(t) + F_{t\to l,6}(t)$]. This signal is then fed to the resonators calculating the instantaneous amplitudes $\xi_k(t)$ of the longitudinal modes. The efficiency can be further increased if those components of the excitation signal F_{res} are not computed where the gain of the longitudinal resonator bank is small.

This model is capable of producing the same sound quality as the model of Sec. V B when the number of resonators implementing the transverse modes equals to the number of string elements in the finite difference model. Figure 6(b) displays the sound pressure spectrum of the first second of a G_1 piano tone synthesized by the resonator-based string model.[25] It can be seen in Fig. 6 that the resonator-based model produces a similar output compared to the composite model of Sec. V B when the string and hammer parameters are set to be the same. The only difference is that the composite string model generates noiselike peaks between the dominant partials due to computational inaccuracies. However, this is not considered as an advantage because the difference between the output of the two models is almost inaudible.[25]

An advantage of this approach is that the computational complexity is reduced to less than the half. Moreover, this method is particularly advantageous when the goal is to reproduce a tone which is similar to that of a particular piano since the measured partial frequencies f_n and decay times τ_n can be directly implemented in the model. On the other hand, the resonator-based model is less physical in the sense that the physical parameters of the string (such as string mass and tension) have only indirect connection to the model.

D. Implications to psychoacoustic research

The informal listening tests made during the development of sound synthesis algorithms raised some questions on the perceptual aspects of longitudinal vibrations. It is an important property of the piano sound that the longitudinal component sounds as an inherent part of the tone, while it is still possible to perceive its pitch.[25] The present authors believe that the reason for this is that the distance of the phantom partials reinforces the pitch information originating from the transverse vibration, while the distance of the ''formant peaks'' leads to the pitch perception of the longitudinal mode. The perceptual effect can be similar to the sound of Touvinian throat- or overtone singers[31] who can produce two tones simultaneously. Listening tests should be conducted to confirm this conjecture.

VI. CONCLUSIONS

The generation of longitudinal components can be summarized as follows: the longitudinal motion is continuously excited by the transverse vibration along the string (and not only during the hammer–string contact). The forced response to this excitation gives a rise to phantom partials, while the free response produces the components corresponding to the longitudinal modal frequencies. Both of these components develop under the assumption of rigid string terminations, i.e., the piano bridge has a less significant effect on the phenomenon.

According to the modal model presented in Sec. III, each longitudinal mode can be viewed as a second-order resonator, whose input is a quasi-harmonic spectrum, containing terms with sum and difference frequencies of some specific transverse modes. As each longitudinal mode emphasizes the peaks around its modal frequency, the sum of their outputs is similar to having formants on a quasi-harmonic spectrum.

In Sec. IV the experimental results of earlier papers have been explained by the results of the modal model, such as why phantom partials originate from adjacent parents and what the inharmonicity coefficient of the phantom partial series is. Some measurements of the present authors have also been outlined, confirming the results of the theoretical model.

Based on the further simplification of the model, two sound synthesis algorithms have been presented in Sec. V. The first one computes the transverse vibration by a finite difference string model and then calculates the inputs of the resonators, which represent the longitudinal modes. The second approach computes both the transverse and longitudinal vibrations in the modal domain, implemented by second-order resonators. Both models produce convincing piano sounds.[25]

The present work has examined the main effects arising from the longitudinal vibration of piano strings. Secondary effects coming from the coupling of longitudinal to transverse polarizations and originating from the characteristics of the string termination could be a subject of future research. For these studies the admittance matrix measurement of the piano bridge would be of great significance. Moreover, the radiation properties of the soundboard as a function of longitudinal string force could also be an interesting field of research. As for sound synthesis models, the computational complexity might be lowered by further simplifications based on psychoacoustic criteria. Accordingly, studies on how the longitudinal components are perceived would also be of great importance.

The findings of the paper can be useful not only in the field of piano acoustics but for researchers interested in the analysis or synthesis of other stringed instruments, such as the guitar. Understanding the generation mechanism of longitudinal components (e.g., which transverse partials excite a specific longitudinal mode) can help piano or guitar builders to achieve a better control over the nature of the tone. For the sound synthesis of other string instruments, the tension computed by the modal model might be used to improve the performance of synthesis models that presently apply the uniform tension approximation.

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APPENDIX: UNIFORM TENSION AS A SPECIAL CASE

The assumption of spatially uniform tension is often applied in the literature (see the work of Legge and Fletcher[11] or Refs. 7–10), since in that case the tension can be computed from the relative elongation of the string by

$$\overline{T}(t)=T_0+ES\frac{1}{2L}\int_{x=0}^{L}\left(\frac{\partial y}{\partial x}\right)^2 dx, \tag{A1}$$

where the string tension $T(x,t)=\overline{T}(t)$ is independent of position x. This assumption is based on the fact that when the speed of the longitudinal waves is much larger than that of the transverse ones, the longitudinal inertial effects can be neglected.[8]

In this case the tension contains terms which have double the frequencies of the corresponding transverse modes.[11] If the transverse displacement is written in a modal form as in Eq. (9), the tension $\overline{T}(t)$ is obtained as

$$\overline{T}(t)=T_0+ES\frac{\pi^2}{4L^2}\sum_{n=1}^{\infty} y_n^2(t)n^2. \tag{A2}$$

It is of some interest to see how the results of this formula can be developed as a special case of the modal model presented in this paper. This both confirms the modal model described in Sec. III and helps in finding the limits where the assumption of uniform tension can be applied.

It can be seen in Eq. (4) that the expression of the tension $T(x,t)$ is made up of three terms. By defining T_0 as the tension at rest, $T_l(x,t)$ as the tension component proportional to the longitudinal slope, and $T_{\mathrm{t}}(x,t)$ as the tension component proportional to the square of the transverse slope, the total tension can be written as

$$T(x,t)=T_0+T_l(x,t)+T_{\mathrm{t}}(x,t). \tag{A3}$$

It the transverse displacement is expressed in the modal form of Eq. (9), the tension component coming from the transverse slope becomes

$$\begin{aligned}T_{\mathrm{t}}(x,t)&=ES\frac{1}{2}\left[\frac{\pi}{L}\sum_{n=1}^{\infty} y_n(t)n\cos\left(\frac{n\pi x}{L}\right)\right]^2\\&=ES\frac{\pi^2}{4L^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} y_m(t)y_n(t)mn\\&\quad\times\left[\cos\left(\frac{m+n}{L}\pi x\right)+\cos\left(\frac{m-n}{L}\pi x\right)\right].\end{aligned} \tag{A4}$$

The Laplace transform of the time domain impulse response $\xi_{\delta,k}(t)$ in Eq. (14c) of a longitudinal mode k is

$$\mathcal{L}\{\xi_{\delta,k}(t)\}=\frac{2}{L\mu}\frac{1}{s^2+(2/\tau_k')s+1/\tau_k'^2+4\pi^2 f_k'^2}, \tag{A5}$$

from which the low frequency response ($s\to 0$ for $f\ll f_k'$) of the resonator can be approximated as

$$\xi_k(t)\approx\frac{2L}{ESk^2\pi^2}F_{\mathrm{t}\to l,k}(t), \tag{A6}$$

which was obtained by writing $f_k'=k\sqrt{ES/\mu}/(2L)$ and assuming $1/\tau_k'\ll f_k'$. The shape of the dashed lines in Figs. 2(a) and (b) confirms that the longitudinal modes have constant gain at low frequencies. Note that this point of the derivation relates to neglecting the inertial effects, i.e., assuming $\mu(\partial^2\xi/\partial x^2)=0$ in Eq. (6). (See also Refs. 8 and 9.)

In this case, the component that comes from the longitudinal motion is expressed as

$$\begin{aligned}T_1(x,t)&=ES\frac{\pi}{L}\sum_{k=1}^{\infty}\xi_k(t)k\cos\left(\frac{k\pi x}{L}\right)\\&=-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}[F_{\mathrm{t}\to l,k}(t)^+\\&\quad+F_{\mathrm{t}\to l,k}(t)^-]\cos\left(\frac{k\pi x}{L}\right).\end{aligned} \tag{A7}$$

Calculating the excitation force $F_{\mathrm{t}\to l,k}(t)=F_{\mathrm{t}\to l,k}(t)^+ + F_{\mathrm{t}\to l,k}(t)^-$ with the help of Eqs. (15a) and (15b) and eliminating k by substituting $m+n=k$ and $|m-n|=k$ gives

$$\begin{aligned}T_1(x,t)=&-ES\frac{\pi^2}{4L^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} y_m(t)y_n(t)mn\cos\left(\frac{m+n}{L}\pi x\right)\\&-ES\frac{\pi^2}{4L^2}\sum_{m=1}^{\infty}\sum_{\substack{n=1\\n\neq m}}^{\infty} y_m(t)y_n(t)mn\cos\left(\frac{m-n}{L}\pi x\right),\end{aligned} \tag{A8}$$

where $n\neq m$ in the second term comes from the fact that the longitudinal mode number $k=|m-n|$ cannot be zero. Note that there is no such constraint for the first term as $k=m+n$ in that case.

If Eqs. (A4) and (A8) are substituted into Eq. (A3), all the terms cancel out, except some with $m=n$ giving

$$T(x,t)=T_0+ES\frac{\pi^2}{4L^2}\sum_{n=1}^{\infty} y_n^2(t)^2 n^2\cos\left(\frac{n-n}{L}\pi x\right), \tag{A9}$$

which is the same as Eq. (A2) obtained by assuming uniform tension along the string. Note that the uniform tension approximation does not lead to zero longitudinal displacement. On the contrary, the string elements move in the longitudinal direction in a way that the tension remains uniform along the string. Actually, the tension could not be uniform without longitudinal motion.

The assumption leading to Eq. (A9) is that all the longitudinal modes are excited by frequencies that are considerably smaller than the corresponding longitudinal modal frequency f'_k. Thus, the validity of the uniform tension approximation should be evaluated by comparing f'_k with the excitation frequencies calculated by Eq. (16) for each k. Having small order transverse vibrations in comparison to the ratio of longitudinal and transverse propagation speeds (see, e.g., Ref. 8) is a sufficient, but not a necessary, condition for the applicability of the uniform tension approximation.

As a special case, if the transverse vibration contains only one mode, the uniform tension approximation can be applied. This is because the transverse mode n excites longitudinal mode $k=2n$, for which Eq. (A6) holds. Note that this is true for all the transverse modes, and not only for the first few. However, when all the transverse partials are present up to a mode number N the assumption of the uniform tension can be applied only if the transverse vibration does not contain significant components around and above $f'_0/2$ (where $f'_0=f'_1$ is the fundamental frequency of the longitudinal modal series). This is because the excitation force $F_{t\rightarrow 1,k}(t)$ has approximately double the bandwidth compared to the bandwidth of the transverse vibration.

It is important to note that the behavior of the longitudinal vibration changes qualitatively as a function of the transverse modes present on the string. It is an interesting field of future research to evaluate the coupling of transverse modes and the properties of nonplanar string vibrations in the case of realistic transverse components, i.e., when the uniform tension approximation cannot be applied. This would answer the question whether the theoretical and experimental results presented for the first few modes of transverse vibrations[7–11] can be applied for the qualitative description of the phenomenon in the case of stringed instruments where dozens of transverse partials are generated.

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