Logarithmic Frequency Scale Parallel Filter Design with Complex and Magnitude-Only Specifications

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***Abstract*—Recently, the fixed-pole design of second-order parallel filters has been introduced to accomplish arbitrary (e.g. logarithmic) frequency resolution for transfer function modeling and equalization. The frequency resolution is set by the pole frequencies, and the resulting filter response corresponds to the smoothed (moving-average filtered) version of the target frequency response. This letter presents the frequency-domain version of the design algorithm for complex and real filter coefficients. The proposed frequency-domain design, besides its computational benefits, allows the use of frequency weighting. In addition, a magnitude-only variation of the algorithm is proposed. Examples of loudspeaker–room modeling and equalization are presented.**

***Index Terms*—IIR digital filters, logarithmic frequency resolution, audio signal processing, loudspeaker–room response equalization.**

I. Introduction

THE problem of modeling or equalizing a given transfer function by a digital filter comes up frequently in the field of digital signal processing. As opposed to the linear frequency resolution of traditional FIR and IIR design algorithms, some tasks require more flexible allocation of frequency resolution. For example, in audio applications logarithmic frequency resolution is desirable, since that corresponds to the resolution of human hearing. For this, various design approaches have been developed, including frequency warped filter design [1], [2] and Kautz filters [3]. Recently, the fixed-pole design of parallel second-order filters has been introduced [4]. The parallel filter requires 33% percent fewer arithmetic operations compared to Kautz filters, while it yields the same transfer function [5].

Filter design (or system identification) is relatively straightforward for low-order systems where the model order can be in the same range as the order of the system. In this case, the poles of the model should correspond to system poles if the optimization procedure was successful. However, when the system order is high (e.g., for a room response it is to the order of hundred thousand), only the overall characteristics can be modeled due to practical limitations in the model order. Traditional system identification methods (like the Prony or Steiglitz-McBride algorithms) lead to poor results because the optimization algorithm will pick and model a few resonant system poles, while most of the other poles are not taken into account. This is illustrated in Fig. 1 (a) for a 32nd order IIR filter designed by the Steiglitz-McBride algorithm (thick line) for modeling a minimum-phase loudspeaker–room response (thin line). While the allocation of frequency resolution is improved when designing a warped IIR filter [1], [2], the problem of modeling only a few resonances still remains, as can be seen in Fig. 1 (b).

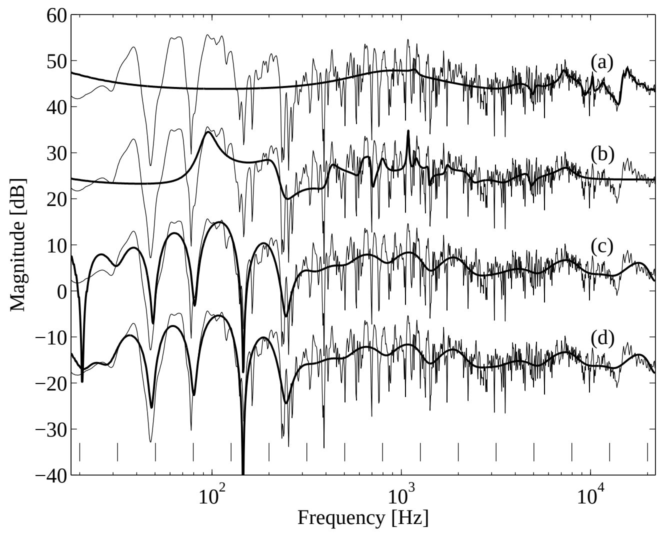

Fig. 1. Filter design comparison. Thick solid lines show the frequency responses of the (a) 32nd order IIR filter, (b) 32nd order warped IIR filter designed with $\lambda = 0.9$, (c) 32nd order parallel filter designed in the time domain, and (d) 32nd order parallel filter designed in the frequency domain. The thin solid lines show the filter specification (minimum-phase loudspeaker–room response) in all cases. The vertical lines indicate the pole frequencies of the parallel filters.

In the case of approximate modeling of high-order systems better results are obtained if the filter poles are predefined according to the desired frequency resolution, and only the zeros are free parameters during parameter estimation. It has been shown in [5] that the fixed-pole design of second-order parallel filters results in a filter response that is similar to smoothing (moving average filtering) the target frequency response. The frequency resolution is defined by the pole-frequency distances [5].

This is illustrated in Fig. 1 (c) where a 32nd order parallel filter is designed using logarithmically spaced pole frequencies from 20 Hz to 20 kHz, indicated by vertical lines in Fig. 1. Now the filter response (thick line) follows the local average

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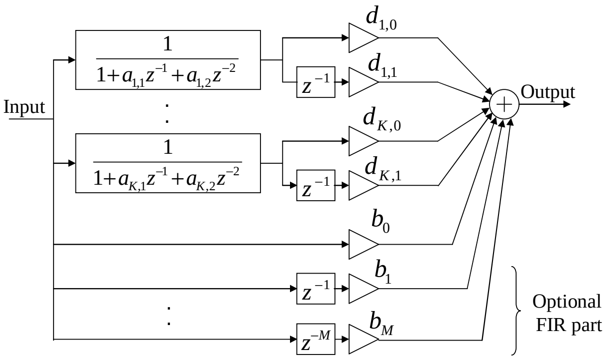

Fig. 2. Structure of the parallel second-order filter.

of the specification (thin line), resulting in a better overall fit compared to IIR or warped IIR filters designed by the Steiglitz-McBride algorithm. Further comparisons to IIR, warped FIR, warped IIR, and Kautz filter designs are given in [4], [5], [6].

This letter presents the frequency-domain version of the fixed-pole parallel filter design algorithm. In Sec. II the least-squares equations are developed for the case of complex and real filter coefficients. In Sec. III a variation of the algorithm is presented, which can be used for magnitude-only specifications. Section IV introduces the use of frequency-dependent weighting and Sec. V presents the frequency-domain direct equalizer design method based on a system-identification approach. Finally, Sec. VI concludes the letter.

II. Basic design algorithm

The general form of the parallel filter consists of a parallel set of second-order sections and an optional FIR filter path:

$$H(z^{-1}) = \sum_{k=1}^{K} \frac{d_{k,0} + d_{k,1}z^{-1}}{1 + a_{k,1}z^{-1} + a_{k,2}z^{-2}} + \sum_{m=0}^{M} b_m z^{-n} \quad (1)$$

where K is the number of second order sections. The filter structure is depicted in Fig. 2.

A. Pole positioning

As the first step of filter design, the pole frequencies f_k are set to a logarithmic frequency scale in the frequency range of interest. For obtaining a $1/\beta$ octave resolution, $\beta/2$ poles are inserted in each octave [5]. Then, the poles of the parallel filter, p_k, are computed using the following formulas [5]:

$$\theta_k = \frac{2\pi f_k}{f_s} \quad (2a)$$

$$p_k = e^{-\frac{\Delta\theta_k}{2}} e^{\pm j\theta_k}, \quad (2b)$$

where θ_k are the pole frequencies in radians given by the predetermined analog frequency series f_k and the sampling frequency f_s. The bandwidth of the kth second-order section $\Delta\theta_k$ is computed from the neighboring pole frequencies

$$\begin{aligned}
\Delta\theta_k &= \tfrac{\theta_{k+1}-\theta_{k-1}}{2} \quad \text{for} \quad k=[2,..,K-1] \\
\Delta\theta_1 &= \theta_2 - \theta_1 \\
\Delta\theta_K &= \theta_K - \theta_{K-1}. \quad (3)
\end{aligned}$$

Equation (2b) sets the pole radii $|p_k|$ in such a way that the transfer functions of the parallel sections cross approximately at their -3dB point (the approximation was obtained by assuming $|p_k| \approx 1$).

B. Weight estimation

Once the denominator coefficients are determined by the poles ($a_{k,1} = p_k + \overline{p}_k$ and $a_{k,2} = |p_k|^2$), the problem becomes linear in its free parameters $d_{k,0}$, $d_{k,1}$ and b_m.

Writing (1) in matrix form for a finite set of ϑ_n angular frequencies yields

$$\mathbf{h} = \mathbf{M}\mathbf{p} \quad (4)$$

where $\mathbf{p} = [d_{1,0}, d_{1,1}, \dots d_{K,0}, d_{K,1}, \ b_0 \dots b_M]^T$ is a column vector composed of the free parameters. The rows of the modeling matrix $\mathbf{M}$ contain the transfer functions of the second-order sections $1/(1+a_{k,1}e^{-j\vartheta_n}+a_{k,2}e^{-j2\vartheta_n})$ and their delayed versions $e^{-j\vartheta_n}/(1+a_{k,1}e^{-j\vartheta_n}+a_{k,2}e^{-j2\vartheta_n})$ for the ϑ_n angular frequencies. The last rows of $\mathbf{M}$ are the transfer functions of the FIR part $e^{-jm\vartheta_n}$ for $m = [0 \dots M]$. Finally, $\mathbf{h} = [H(\vartheta_1) \dots H(\vartheta_N)]^T$ is a column vector composed of the resulting frequency response.

Now the task is to find the optimal parameters $\mathbf{p}_{\mathrm{opt}}$ such that $\mathbf{h} = \mathbf{M}\mathbf{p}_{\mathrm{opt}}$ is closest to the target frequency response $\mathbf{h}_{\mathrm{t}} = [H(\vartheta_1)_{\mathrm{t}} \dots H(\vartheta_N)_{\mathrm{t}}]^T$. If the error is evaluated in the mean squares sense

$$e_{\mathrm{LS}} = \sum_{n=1}^{N} |H(\vartheta_n) - H(\vartheta_n)_{\mathrm{t}}|^2 = (\mathbf{h}-\mathbf{h}_{\mathrm{t}})^H(\mathbf{h}-\mathbf{h}_{\mathrm{t}}), \quad (5)$$

the minimum of (5) is found by the well-known least-squares (LS) solution

$$\mathbf{p}_{\mathrm{opt}} = (\mathbf{M}^H\mathbf{M})^{-1}\mathbf{M}^H\mathbf{h}_{\mathrm{t}} \quad (6)$$

where $\mathbf{M}^H$ is the conjugate transpose of $\mathbf{M}$.

Note that (6) assumes a filter specification $H_{\mathrm{t}}(\vartheta_n)$ given for the full frequency range $\vartheta_n \in [-\pi, \pi]$. Thus, the design can be used for obtaining filters with complex coefficients, since the frequency specification is not constrained to be conjugate-symmetric.

However, in most of the cases we are interested in filters with real coefficients: in this case either the user has to ensure that $H_{\mathrm{t}}(-\vartheta_n) = \overline{H}_{\mathrm{t}}(\vartheta_n)$, where $\overline{H}_{\mathrm{t}}$ is the complex conjugate of H_{t}, or, in the case of one sided ($\vartheta_n \in [0, \pi]$) specifications, the following formula has to be used instead of (6):

$$\mathbf{p}_{\mathrm{opt}} = (\mathrm{Re}\{\mathbf{M}^H\mathbf{M}\})^{-1}\mathrm{Re}\{\mathbf{M}^H\mathbf{h}_{\mathrm{t}}\} \quad (7)$$

where $\mathrm{Re}\{\mathbf{A}\}$ corresponds to taking the real part of $\mathbf{A}$.

C. Comparison to time-domain filter design

The time-domain and frequency-domain versions of parallel-filter design provide the same result if the ϑ_n frequencies are distributed evenly according to a linear frequency scale and the grid is dense enough. This is due to Parseval's theorem: if the energy of the estimation error is minimal in the time-domain, so is it in the frequency-domain. However, if the ϑ_n frequencies are given at a logarithmic frequency scale, slightly different results are achieved, as displayed in Fig. 1 (d) for a 16 section (32nd order) design. Since the measured loudspeaker response was available on a linear frequency scale, a 128 point logarithmic frequency scale specification was computed by averaging the neighboring linear frequency bins (not displayed in Fig. 1). The slight improvement compared to the time-domain method (c) visible at low frequencies arises because now the error is minimized on a logarithmic frequency grid as opposed to the linear-grid equivalent of the time-domain design.

If the filter specification $H_{\mathrm{t}}(\vartheta_n)$ is available in the frequency domain, as often the case, a benefit of the frequency-domain approach compared to the time-domain design [4] is that the impulse response does not have to be computed. Note that computing the impulse response does not only involve IFFT operation, but also a suitable interpolation of the frequency response, if the frequency points are not spread uniformly along the unit circle.

In addition, in the case of a non-uniform (e.g. logarithmic) frequency specification, the frequency-domain design requires fewer specification points (shorter $\mathbf{M}$ matrix and $\mathbf{h}_{\mathrm{t}}$ vector) compared to the time-domain version, leading to lower computational complexity for filter design. This is because the length of the corresponding time-domain target response is defined by the minimal frequency distance in the frequency-domain specification. In the case of a logarithmic frequency scale specification, the resulting target impulse response will have a long low-frequency tail corresponding to the high resolution required at low frequencies. For example, for the designs of Fig. 1 (c) and (d) the frequency-domain method required 80 times fewer specification points and thus around two orders of magnitude smaller design time compared to the time-domain design. The reduced complexity may be beneficial for certain real-time applications, such as loudspeaker–room equalization based on on-line measurements.

III. Magnitude-only filter design

Oftentimes only the magnitude of the target frequency response is specified, and the phase of the filter can be arbitrary. In this case the magnitude error

$$e_{\mathrm{magn}} = \sum_{n=1}^{N} (|H(\vartheta_n)| - |H(\vartheta_n)_{\mathrm{t}}|)^2, \tag{8}$$

should be minimized instead of the complex transfer function error of (5). For this, the above design algorithm is modified based on the iterative technique originally presented for the frequency-domain Steiglitz-McBride algorithm [7]. The method is based on the fact that minimizing the complex transfer function error of (5) corresponds to magnitude error minimization if the phase of the filter $\varphi\{H(\vartheta_n)\}$ and the specification $\varphi\{H_{\mathrm{t}}(\vartheta_n)\}$ are equal, since in this case we have $|H(\vartheta_n) - H_{\mathrm{t}}(\vartheta_n)|^2 = (|H(\vartheta_n)| - |H_{\mathrm{t}}(\vartheta_n)|)^2$.

As a starting point, a minimum-phase target specification $H_{\mathrm{t},0}(\vartheta_n)$ is obtained from the magnitude specification based on the Hilbert-transform relation of magnitude and phase of minimum-phase transfer functions [8]. Then, the filter coefficients are estimated according to (6) or (7).

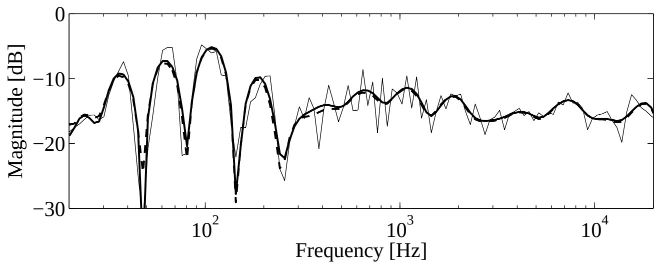

Fig. 3. Magnitude-only parallel filter design: minimum-phase specification (thin line), magnitude response of the first filter $|H_1(\vartheta)|$ (thick dashed line), and the final filter magnitude response $|H_{10}(\vartheta)|$ after 10 iterations (thick solid line).

Next, an iterative procedure is started where the phase of the specification is adjusted to match the phase of the filter obtained in the previous step $\varphi\{H_{\mathrm{t},i}(\vartheta_n)\} = \varphi\{H_{i-1}(\vartheta_n)\}$, while the magnitude is kept unchanged. A new filter $H_i(\vartheta_n)$ is designed based on this updated specification. The convergence of the procedure is fast, requiring five-ten iterations in practice.

Figure 3 shows a 32nd order parallel filter design. The target is a loudspeaker–room response given by 128 specification points on a logarithmic scale (thin line in Fig. 3). It can be seen that the fit improves only slightly by the iterations (thick solid line) compared to the first filter (thick dashed line), showing that the minimum-phase target is a good starting point for the magnitude-only design.

IV. Frequency dependent weighting

A further benefit of designing the fixed-pole parallel filter in the frequency-domain is that this allows adding different weights to the different specification points.

In this case, the error becomes

$$e_{\mathrm{WLS}} = \sum_{n=1}^{N} W(\vartheta_n)|H(\vartheta_n) - H(\vartheta_n)_{\mathrm{t}}|^2 = (\mathbf{h} - \mathbf{h}_{\mathrm{t}})^H \mathbf{W} (\mathbf{h} - \mathbf{h}_{\mathrm{t}}), \tag{9}$$

where $W(\vartheta_n)$ is the weight for the ϑ_n frequency, and $\mathbf{W}$ is the weighting matrix having $W(\vartheta_n)$ in its diagonal and zeros elsewhere. The minimum is obtained by the weighted-least-squares (WLS) solution:

$$\mathbf{p}_{\mathrm{opt}} = (\mathbf{M}^H \mathbf{W} \mathbf{M})^{-1} \mathbf{M}^H \mathbf{W} \mathbf{h}_{\mathrm{t}}, \tag{10}$$

or, in the case of one sided ($\vartheta_n \in [0, \pi]$) specifications:

$$\mathbf{p}_{\mathrm{opt}} = (\mathrm{Re}\left\{\mathbf{M}^H \mathbf{W} \mathbf{M}\right\})^{-1} \mathrm{Re}\left\{\mathbf{M}^H \mathbf{W} \mathbf{h}_{\mathrm{t}}\right\} \tag{11}$$

For example, if the target specification was obtained by averaging multiple noisy responses and the variance σ_n^2 was also computed, it is possible to use weighting so that the less reliable data points have a smaller effect in the error to be minimized. By using $W(\vartheta_n) = 1/\sigma_n^2$ the best linear unbiased estimator (BLUE) of the measured system response is obtained for a given pole set, which is equivalent to the maximum-likelihood estimate in the case of Gaussian measurement noise.

V. Direct Equalizer Design by a System Identification Approach

Equalizing a system by the parallel filter can be done by dividing the desired target response $H_{\mathrm{t}}(\vartheta_n)$ (e.g. a bandpass response) by the system response $H_{\mathrm{s}}(\vartheta_n)$ and designing a parallel filter for this $H_{\mathrm{t}}(\vartheta_n)/H_{\mathrm{s}}(\vartheta_n)$ specification according to Sec. II. However, the narrow dips of $H_{\mathrm{s}}(\vartheta_n)$ result in sharp peaks in $H_{\mathrm{t}}(\vartheta_n)/H_{\mathrm{s}}(\vartheta_n)$ because of the division, biasing the filter design.

While the problems of division can be reduced by regularization, a more appropriate way of designing an equalizer is to minimize the error between the final, equalized response $H_{\mathrm{eqd}}(\vartheta_n)$ and the target frequency response $H_{\mathrm{t}}(\vartheta_n)$, as was also proposed in the case of time-domain design in [4]. This is basically a system identification problem with output error minimization: the input of the parallel filter is the system response $H_{\mathrm{s}}(\vartheta_n)$ and we should estimate the filter parameters such that its output $H_{\mathrm{eqd}}(\vartheta_n)$ best matches the target response $H_{\mathrm{t}}(\vartheta_n)$.

Accordingly, the equalized response is given by

$$H_{\mathrm{eqd}}(z^{-1}) = H(z^{-1})H_{\mathrm{s}}(z^{-1}) = \sum_{k=1}^{K} \frac{d_{k,0} + d_{k,1}z^{-1}}{1 + a_{k,1}z^{-1} + a_{k,2}z^{-2}} H_{\mathrm{s}}(z^{-1}) + \sum_{m=0}^{M} b_m z^{-n} H_{\mathrm{s}}(z^{-1}). \tag{12}$$

Writing this in a matrix form for a finite set of ϑ_n angular frequencies yields

$$\mathbf{h}_{\mathrm{eqd}} = \mathbf{M}_{\mathrm{eq}}\mathbf{p}_{\mathrm{eq}} \tag{13}$$

where $\mathbf{p}_{\mathrm{eq}} = [d_{1,0}, d_{1,1}, \ldots d_{K,0}, d_{K,1},\ b_0 \ldots b_M]^T$ is a column vector composed of the free parameters of the parallel equalizer. The rows of the equalizer modeling matrix $\mathbf{M}_{\mathrm{eq}}$ are obtained from the modeling matrix $\mathbf{M}$ constructed in Sec. II by multiplying them with the system frequency response $H_{\mathrm{s}}(\vartheta_n)$. For example, instead of $1/(1 + a_{k,1}e^{-j\vartheta_n} + a_{k,2}e^{-j2\vartheta_n})$ we simply have $H_{\mathrm{s}}(\vartheta_n)/(1 + a_{k,1}e^{-j\vartheta_n} + a_{k,2}e^{-j2\vartheta_n})$. Finally, $\mathbf{h}_{\mathrm{eqd}} = [H_{\mathrm{eqd}}(\vartheta_1) \ldots H_{\mathrm{eqd}}(\vartheta_N)]^T$ is a column vector composed of the resulting final frequency response. Since (13) has the same structure as (4), the optimal set of parameters are obtained in the same way as in Sec. II, e.g. by (6) or (7) for complex and real filters, respectively.

The comparison of division-based and direct equalizer design is presented in Fig. 4. The equalization based on transfer function division (b) leads to biased results due to the inverted dips of the system response, while the direct equalizer design (c) does not show this erroneous behavior. The pole frequencies (vertical lines in Fig. 4) were chosen to have higher density at low frequencies for increased resolution in the more problematic region of the transfer function.

Note that magnitude-only design and frequency-dependent weighting can be used also for direct equalizer design in the same way as presented for filter design in Secs. III and IV.

VI. Conclusion

This letter has presented the frequency-domain variant of the fixed-pole second-order parallel filter design algorithm. The new method allows the use of magnitude-only specifications and frequency-dependent weighting, which is also useful for taking into account the different reliability of the specification points. Finally, frequency-domain direct equalizer design by the system identification approach was presented and compared to equalizer design based on transfer function division. Matlab code for parallel filter design is available at http://www.mit.bme.hu/∼bank/parfilt.

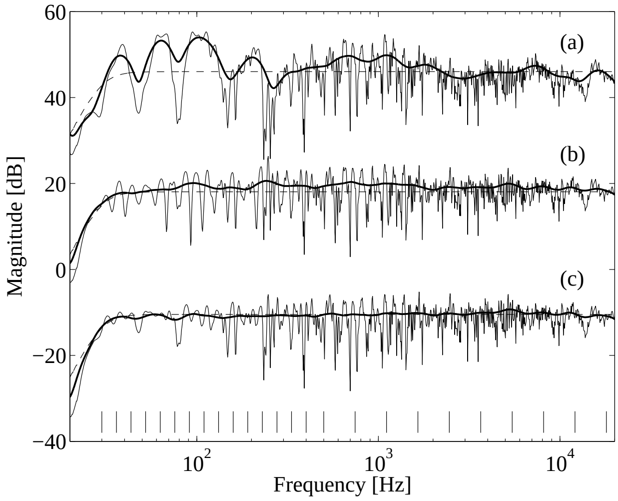

Fig. 4. Minimum-phase room response equalization: (a) unequalized loudspeaker–room response, (b) equalized by a 50th-order parallel filter designed for division based $H_{\mathrm{t}}(\vartheta_n)/H_{\mathrm{s}}(\vartheta_n)$ specification, and (c) equalized by a 50th-order parallel filter estimated by the direct equalizer design of Sec. V. The thick lines show the third-octave smoothed versions of the transfer functions, and the target specification is displayed by dashed lines. The same pole frequencies were used for both filter designs, indicated by vertical lines.

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