MODELING THE LONGITUDINAL VIBRATION OF PIANO STRINGS

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ABSTRACT

This study is motivated by the physical modeling of the longitudinal string vibrations in the piano. Informal listening tests show that the longitudinal vibrations play an important role in the attack of the sound, and are responsible for the metallic character of low notes. First, a simple mathematical model is developed for qualitative understanding. Detailed analysis is given for sinusoidal transversal displacement with non-rigid termination, clarifying the generation of phantom partials. To investigate how these effects develop in more natural circumstances, finite-difference string and hammer models are used, with parameters taken from real pianos. For real-time sound synthesis purposes, an efficient modeling approach is presented. The model extends the digital waveguide string model by implementing two additional string models, one for the phantom partials and one for the longitudinal modes.

1. INTRODUCTION

Physical modeling of the piano has been an interesting field of research in the last decades, see, e.g. [1, 2, 3]. As faster processors emerge, real-time implementations become possible. However, the quality of synthesized piano sound is still far from perfect. The digital waveguide [4] used in these models is capable of producing a quasi-periodic sound, which is built up by exponentially decaying sinusoids. Conversely, when the spectra of a real piano sound is observed, other components can also be noticed. These correspond to either the longitudinal modes of the string [5], or to the "phantom" partials generated by nonlinear mixing [6]. We believe that these two phenomena should be treated together.

The motivation of this research was to refine the quality of our piano model. However, as no detailed analysis can be found in the literature on the generation of these phenomena, first the underlying physics has to be understood before efficient physical models are developed. The paper is organized as follows: first, prior work is presented and the basic properties of phantom partials and longitudinal modes are described. This is followed by the analysis of steady-state motion and by the investigation of the transient response by a numerical model. Emphasis is given on the explanation of phantom partials. For real-time sound synthesis, an efficient algorithm is presented by applying digital waveguide string models.

2. PRIOR WORK

By observing the spectrum of piano sound, "phantom" partials can be found between the normal, inharmonic partial series [6]. They are generated by nonlinear frequency mixing, thus, their frequencies can be computed as the sum or difference of the normal partial frequencies. Those, which appear at the double frequency of a normal partial, called "even" phantoms. Accordingly, "odd" phantoms appear at the sum or difference of two different partial frequencies. In a perfectly harmonic instrument, these sum and difference frequencies will correspond to the frequencies of "normal" partials, thus, phantom partials do not influence the sound significantly. However, when the transversal vibration has an inharmonic frequency series, which is the case for the piano, phantom partials will depart from the normal partials. As measurements show, odd phantoms are generally produced by adjacent parents (e.g., $5 + 6$, rather than $4 + 7$) [6]. That paper does not describe the reason of this fact, nor the detailed mechanism how these phantoms arise.

Somewhat earlier, a second series (the "lower series") of partials were extracted from the spectrum of a piano in [7]. This lower series of partials has a lower inharmonicity. The inharmonicity coefficient B has been found to be around the fourth of the one for normal partials. Explanation of this fact is not given in the paper. Moreover, the authors argue that the lower series is possibly generated by the string vibration parallel to the soundboard. However, there is no physical reason why the string should have a different dispersion in the two transversal polarizations. We believe that the "lower series" is equivalent with the phantom partials generated by nonlinear mixing.

Some phantom partials are displayed in Fig. 1 for an $A_4^\sharp$ note with circles. The marked phantoms appear at the frequencies in terms of the normal partial series at $f_4 + f_5$, $2f_5$, $f_6 + f_7$, and $f_7 + f_8$. These phantom partials may play a part in differentiating the timbre between pianos by emphasizing the effect of inharmonicity with the beats produced between them and the normal partials [6].

The longitudinal modes of the piano string may have a more significant perceptual effect. In the low range of the piano, the pitch of these components can be perceived by the listener, and the subjective quality of the instrument is highly dependent on the frequency of these modes [5]. The longitudinal modes are a nonlinear function of the transversal vibration, which justifies the assumption that they are excited by the string stretching due to a transversal vibration of a finite amplitude [8].

3. THE BASIC EQUATION

A real piano string is vibrating in two transversal planes, and in the longitudinal direction as well. Principally, piano hammers excite one polarization of the string, the other two are gaining energy through coupling. Throughout the vibration, these polarizations interact with each other, as a result of nonlinear behavior of the string.

For simplicity, let us assume that the string is vibrating in one plane, thus, one transversal and one longitudinal polarization is

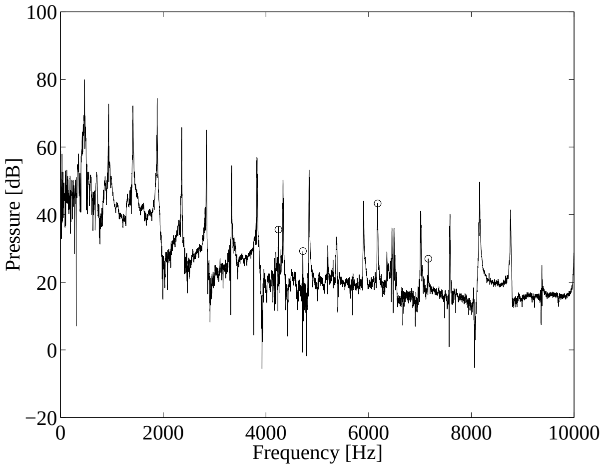

Figure 1: *Spectrum of an $A_4^\sharp$ piano tone with some phantoms marked by circles.*

present. When there is a transversal displacement on the string, the string elongates. This results in a force exciting a longitudinal wave in the string. The longitudinal wave modulates the tension along the string, which influences the transversal vibration. As we are interested in the generation of longitudinal waves, a further assumption is made: the influence of the longitudinal polarization on the transversal one is neglected.

The equations describing the interaction of the two transversal and the longitudinal polarizations are developed e.g., in [9]. After some approximations, the longitudinal motion can be characterized by the following equation:

$$\mu \frac{\partial^2 \xi}{\partial t^2} = ES \frac{\partial^2 \xi}{\partial x^2} + \frac{1}{2} ES \frac{\partial \left(\frac{\partial y}{\partial x}\right)^2}{\partial x} \tag{1}$$

where $y = y(x,t)$ and $\xi = \xi(x,t)$ are the transversal and longitudinal displacement of the string with respect to time t and space x. The mass per unit length is referred by μ, E is the Young's modulus and S is the cross-section area of the string. Eq. (1) is the standard wave equation with an additional force term depending on the transversal vibration of the string, according to a second-order nonlinearity.

4. STEADY STATE SOLUTION

By applying the above equations, the coupling between the transversal and longitudinal string vibration can be computed analytically for sinusoidal transversal vibration. This corresponds to describing the origin of phantom partials of [6] or the "lower series" of [7].

4.1. Sinusoidal transversal wave

Considering one single transversal mode of a lossy string with rigid terminations, the transversal string displacement will be [9]:

$$y(x,t) = A_k \sin(2k\pi f_0 t) e^{-\frac{t}{\tau_k}} \sin(\frac{k\pi}{L} x) \tag{2}$$

where A_k stands for the initial amplitude, and τ_k for the decay time of mode k. The length of the string is referred by L, and f_0 is the fundamental frequency, i. e., the frequency of mode $k = 1$. According to Eq. (1), the longitudinal excitation force-distribution $F_{l,e}(x,t)$ can be obtained as follows:

$$F_{l,e}(x,t) = ES \frac{k^3 \pi^3}{4L^3} A_k^2 \left(\cos(4k\pi f_0 t) - 1\right) e^{-\frac{2t}{\tau_k}} \sin(\frac{2k\pi}{L} x) \tag{3}$$

From this, some interesting observations can be made. The force can be separated to a decaying cosine and to a decaying static force. The amplitude of the force $F_{l,e}(x,t)$ is proportional to the square of initial transversal displacement A_k, the decay time is the half of τ_k, and the frequency of force variation is the double of the transversal vibration frequency kf_0. Concerning the spatial distribution of $F_{l,e}(x,t)$, a transversal mode k produces an excitation force corresponding to a longitudinal mode $k_l = 2k$.

The normal modes of the longitudinal vibration can be written similarly to Eq. (2). Now the question is if $F_{l,e}(x,t)$ can excite the longitudinal mode k_l of the string. For efficient coupling, two conditions has to be met: the spatial distribution of $F_{l,e}(x,t)$ should not be orthogonal to the longitudinal-mode shape $\sin(\frac{k_l \pi}{L} x)$, and the frequency of the excitation $2kf_0$ should be close to that of the longitudinal mode $k_l f_{l,0}$. Coming from the first condition, a transversal mode k can only excite a longitudinal mode $k_l = 2k$. Thus, a force with a frequency of $2kf_0$ should excite a longitudinal mode with a frequency of $2kf_{l,0}$. As $f_{l,0}$ is of an order higher than f_0 in practice, this results in a small longitudinal motion only, although it might reach the air through the soundboard.

In reality, the string is terminated with a finite impedance. Therefore, the normal modes do not have a node at the termination. This can be taken into account in Eqs. (2) and (3) by substituting L with $L + \delta L$, while $0 \le x \le L$ still holds. We can still assume that the termination is perfectly rigid in the longitudinal direction. The result will be that the force $F_{l,e}(x,t)$ is not orthogonal to any of the longitudinal mode shapes, thus, it excites all of them, although in a different way. If the frequency of the excitation is near to a longitudinal-mode frequency, i.e., $2kf_0 \approx k_l f_{l,0}$, strong longitudinal vibration emerges. For example, if $f_{l,0} \approx 10 f_0$, then the fifth transversal mode can excite the first longitudinal mode strongly. This is in a good accordance with the measurements of [6], where some cases phantom partials with frequencies near to the longitudinal mode have produced larger sound pressure than the neighboring transversal ones.

As the excitation force of Eq. (1) is known, the corresponding longitudinal vibration could be analytically calculated by the help of the Green's function [9]. However, as the role of this section is to help qualitative understanding, these derivations are not included in the paper.

4.2. Mode pairs

In practice, the transversal vibration is made up of several modes. Hence, the excitation force $F_{l,e}(x,t)$ will contain terms with the sum and difference frequencies of all the transversal modal frequencies, coming from the second-order nonlinearity of the coupling. When N normal transversal partials are present, they generate N^2 phantom partials. However, these are not all noticeable in the spectrum: generally the phantoms produced by transversal parents with consecutive mode numbers (e.g., $5 + 6$) appear [6]. Let us examine the reason in detail.

As the nonlinearity is of second order, it is enough to explain the phenomena for the sum of two sinusoids. When the excitation force $F_{l,e}(x,t)$ is calculated for the sum of two exponentially decaying sinusoids described by Eq. (2), the force distribution $F_{l,e}(x,t)$ with the sum and difference frequencies will be the following:

$$F_{l,e}(x,t) = -ES\frac{k_1 k_2 \pi^3}{4L^3} A_1 A_2 F_{l,e}(t) \times$$
$$\times \left(\sin(\tfrac{k_1+k_2}{L}\pi x)(k_1+k_2) + \sin(\tfrac{k_2-k_1}{L}\pi x)(k_2-k_1)\right) \quad (4)$$

where $F_{l,e}(t)$ is the time dependent component containing two cosine functions with the sum and difference frequencies of transversal mode k_1 and k_2. Note that Eq. (4) does not include the double-frequency terms for clarity (they are the same as for the single sine case of Eq. (3)).

We have seen that a longitudinal mode k_l is significantly excited only if the spatial distribution of $F_{l,e}(x,t)$ is not orthogonal to the longitudinal mode shape $\sin(\frac{k_l \pi}{L}x)$, and the excitation frequency is close to the modal frequency $k_l f_{l,0}$. The sum frequency of the lower (e.g., $k < 10$) transversal modes are around the frequency of the first longitudinal mode $k_l = 1$. Accordingly, those $k_1 + k_2 = m$ combinations will be present in the longitudinal motion, where $k_2 - k_1 = 1$, since in the spatial distribution of $F_{l,e}(t)$ the $\sin(\frac{\pi}{L}x)$ term will be present. In the case of higher phantom partials, higher longitudinal modes should be excited, thus, the parents will be transversal modes with mode numbers $k_2 - k_1 > 1$.

It has to be noted that a phantom generated by modes $k_2 - k_1 = 1$ appears almost at the same frequency where the one produced by mode $k_2 - k_1 = 3$ appears, if $k_1 + k_2 = m$ is the same for the two case. This is because the sum frequencies produced by the frequency pairs $k_1 + k_2$ are not distributed evenly on the frequency axis, but have a larger density around the center frequency determined by $k_2 - k_1 = 1$ (the deviation from this center frequency is proportional to $(k_2 - k_1)^2$).

By knowing that odd phantoms are mainly produced by adjacent parents, and that even phantoms can be found at the double frequencies of normal partials, we can easily explain why the inharmonicity coefficient B is the $1/4$ part of the value for normal partials. We can calculate the frequencies f_m of even phantoms $m = 2k$ easily, by the help of the inharmonicity formula of [10]:

$$f_m = 2f_k = 2f_0 k\sqrt{1+Bk^2} = f_0 m\sqrt{1+\frac{1}{4}Bm^2} \quad (5)$$

For even phantoms, the expression is accurate. For odd phantoms, $k = m/2$ is not an integer number, but Eq. (5) can be still applicable, since we can assume that the inharmonicity curve is smooth enough.

These theoretical results were justified by a finite difference model based on Eq. (1) with losses added. The transversal vibration has been analytically computed, which have made it easy to experiment with diffrerent transversal mode numbers and vibration frequencies. For low transversal mode numbers ($k_1, k_2 < 10$) adjacent parents ($k_2 - k_1 = 1$) produced 15-20 dB higher longitudinal motion compared to other combinations of $k_1 + k_2 = m$.

Those sum terms which do not form a phantom partial ($k_2 - k_1 \gg 1$), together with the difference components, form a dense, noise-like excitation spectrum with thousands of partials. These are measured in the sound spectrum as broadband noise. However, when a frequency of a term is close to that of a longitudinal mode, it can excite that mode effectively. As the excitation force spectrum is dense, it is very probable that the longitudinal modes of the string will be excited throughout the vibration. Note that for a perfectly harmonic instrument the excitation force spectrum would contain some distinct peaks, exciting the longitudinal modes only in special constellations of the longitudinal and transversal fundamental partial frequencies.

5. TRANSIENT RESPONSE

Up to know, only the phenomenon of phantom partials has been considered, which is the forced motion of the longitudinal modes. However, as the excitation in a real piano is not steady-state, the free vibration of the longitudinal modes will also appear.

A finite-difference string model has been realized for calculating the transversal vibration with hammer excitation. The parameters of the model were taken from [11]. The transversal displacement produced by the model serve as an input for the finite-difference longitudinal string model. The longitudinal string model is based on Eq. (1), with losses added. Both in the transversal and longitudinal string models 100 elements were used. A high sampling frequency ($f_s = 441$ kHz) was necessary to maintain the numerical stability of the longitudinal string model.

The output of the model has shown that the longitudinal vibrations excited by the hammer strike are the most significant components of the longitudinal motion. Figure 2 displays the spectrum of the force at the bridge in the direction of the strings for a C_2 piano string, with the parameters of [11]. The Young's modulus E and the cross section area of the string S were set according to values found in real pianos. The impact velocity of the hammer was set to 5 m/s.

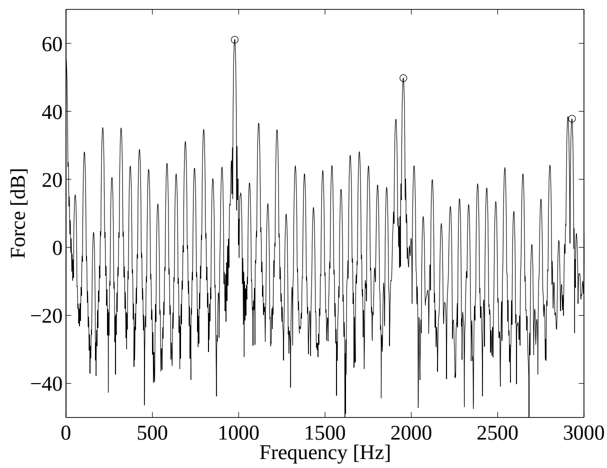

Figure 2: *Force spectrum of the longitudinal vibration of a C_2 note. The longitudinal modal frequencies are marked by circles.*

The components of the free longitudinal vibration are marked by circles in Fig. 2. It can also be seen that around these frequencies the phantom partials are emphasized, producing beating with the longitudinal modes. Note that in real pianos the longitudinal modes are not in a perfect harmonic series, and simulating them in this way produces poor sonic results. Further research is needed to determine whether the "inharmonicity" of the longitudinal modes are produced by the finite impedance or caused by the properties of the string.

6. THE ROLE OF FEEDBACK

In some cases the simulation produced unexpected results, where the longitudinal vibration had very large amplitudes. This happens when a phantom partial lies very close to a longitudinal mode, exciting it with its eigenfrequency. This cannot occur in reality, since the transversal modes loose energy when they excite the longitudinal motion. When the effect of the longitudinal vibration on the transversal one is also taken into account, the same longitudinal modes behave as expected. The wave equation for the transversal motion can be written by approximating the formulas of [9]:

$$\mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2} + ES \frac{\partial \left(\frac{\partial y}{\partial x} \frac{\partial \xi}{\partial x} \right)}{\partial x} \qquad (6)$$

It can be seen that the components coming from the longitudinal motion are the sum and difference frequencies of one transversal and one longitudinal mode. Let us assume that the transversal modes $k_1 + k_2$ excite a longitudinal mode strongly with a frequency $f_{l,k} = f_0(k_1 + k_2)$. Now the difference frequencies between the longitudinal and transversal modes can react with the other modes of the mode pair k_1 and k_2, since $f_{l,k} - f_0 k_1 = f_0 k_2$ and $f_{l,k} - f_0 k_2 = f_0 k_1$. Further investigations are required to understand how this feedback can diminish the energy of the modes k_1 and k_2.

7. THE DIGITAL WAVEGUIDE MODEL

A finite-difference string model would require unacceptable computational costs for real-time sound synthesis. A highly efficient string modeling technique, the digital waveguide modeling [4] reduces this complexity by an order of 100. However, implementing the interaction of the longitudinal and transversal polarizations is not a straightforward task.

One option is calculating the transversal vibration by a digital waveguide model, and running a finite-difference longitudinal string model in parallel with the excitation force calculated from the transversal displacement. The required computational complexity can be reduced by implementing only 5-10 string elements in the longitudinal model, since the task is to simulate the first few longitudinal modes. The longitudinal vibration must be made slightly inharmonic by e.g., applying inhomogeneous string parameters. This model produces acceptable quality at a reasonable computational cost, but in some cases the longitudinal vibration can be unexpectedly large, similarly as described in Sec. 6. Feedback from the longitudinal vibration to the transversal one is not an option here, since the stability of the system cannot be maintained because of the large number of approximations.

A simpler, but less physical solution is implementing the phantom partials and the free vibration of the longitudinal modes directly with two other string models. As we have seen in Sec. 4.2, the phantom series has a similar inharmonicity curve as the normal partials, but with inharmonicity coefficient $B/4$. This can be easily realized with a second digital waveguide in parallel. Moreover, informal listening tests show that this second waveguide can be perfectly harmonic, so there is no need for a high-order dispersion filter. To render the nonlinear dependence of the amplitudes of phantom partials on the strike velocity, the hammer force calculated by the basic string model is squared before lead to the phantom string model. The longitudinal vibrations are simulated in a similar way, but in that case a perfectly harmonic waveguide produces poor results. Therefore, either an allpass filter is needed to alter the modal frequencies, or the small number of required partials are modeled by a parallel resonator bank. The model produces good sound quality, although the longitudinal modes sound to be separated from the main tone in some cases. This can be overcome by increasing the number of partials to produce an inharmonic tone, thus, a less definite pitch.

8. CONCLUSIONS

We have described the phenomenon how the longitudinal vibration of a string is excited by the transversal vibration with a simplified mathematical model. The longitudinal motion of a string is made up of the free vibration of the longitudinal modes and the forced vibration coming from the transversal displacement with a quadratic nonlinearity. The generation mechanism of phantom partials was described in detail. The theoretical results were verified by computer simulations with a finite-difference string model. For sound synthesis purposes an efficient digital waveguide based solution has been presented.

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