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Exercises in Measurement Technology

for Electrical Engineering Students

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This work consists of selected items of the original textbook, Sujbert, L., L. Naszádos, and G. Péceli, "*Méréstechnika példatár villamosmérnököknek*", Műegyetemi Kiadó, Budapest, 2006, id. number: 55078, in Hungarian.

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Preface

This booklet is based on the Hungarian textbook "*Méréstechnika példatár villamosmérnököknek*" published in 2006. Here only those items are presented which are frequently used in the Measurement Technology classes.

The first part of the booklet contains the problems, while the second part presents the solutions. Many of the solutions are much more detailed than the original Hungarian version to help understanding. The difficulty level of the problems increase in each chapter, therefore it is advised to solve them in a consecutive order.

The Authors intend to draw the attention of the Reader that practicing using only this booklet cannot substitute the careful learning and understanding of the theoretical basics of measurement technology.

Budapest, June 2014 *László Sujbert*

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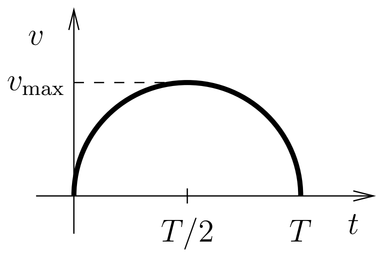
Part I

Problems

Chapter 1

Basic problems

1.1.[1]



A model car is tested on a straight road. The car first accelerates to v_{max} velocity, then stops. Since the car was designed by engineers, its velocity looks like a semicircle, as can be seen in the figure above. What is the total distance taken by the car if $v_{\text{max}} = 40$ km/h, and the elapsed time is $T = 30$ s?

1.2. Determine the SI unit of the below quantities and functions assuming that the unit of the original signal is volt (V) as a function of the time (s). (Note that some of the below quantities may not exist for a specific signal. SI: Système International; International system of units)

a) signal power;

b) Fourier-transform;

c) correlation;

d) power density spectrum;

e) energy density spectrum;

f) effective value;

g) RMS-value;

h) variance;

[1] J. B. Csernyak, R. M. Rose, *"A minszki csirke és további 99 elgondolkodtató feladat az orosz matematikai és fizikai hagyományokból", in Hungarian*

i) mean square value;

j) expected value;

k) standard deviation.

1.3. A system is excited by voltage, the output is current. What is the SI unit of the impulse response of the system?

1.4. Determine the complex Fourier-series of the $x(t) = A\cos(2\pi ft + \varphi)$ signal!

1.5. Determine the complex Fourier-series of the $x(t) = A_1\cos(2\pi f_a t) + A_2\cos(5\pi f_a t)$ signal! What is the time period of this signal?

1.6. Is the following signal periodic?

$$x(t) = A_1 A_2 \sin(2\pi f_1 t)\sin(2\pi f_2 t), \quad f_2 = 1.6 f_1.$$

If yes, what is the period time?

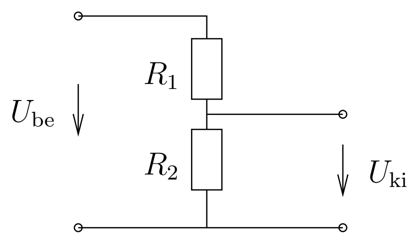
Chapter 2

Error calculation I.

2.1. Velocity is measured by the measurement of time and displacement. The measured value of the displacement is $x = 2000$ m $\pm$ 0.5%, and the measured time is $t = 2000$ s$\pm$0.1%. What is the worst case error of the estimated velocity?

2.2. 100 resistors of 1 kΩ nominal value and 1% tolerance (relative random error) are connected in series. What is the relative error of the resulting resistance having 100 kΩ nominal value, using the **(a)** *worst case* and **(b)** the *probabilistic* summation of error components?

2.3.



The figure displays a voltage divider made up of two resistors. The values of the resistors are $R_1 = 49$ kΩ and $R_2 = 1$ kΩ. The tolerance of both resistors is 100 ppm.

a) What is the nominal value of the voltage division ratio?

b) What is the worst case relative error of the voltage divider?

c) Shall we consider the error of the divider as a systematic or random error, if **(1)** we have a company which produces voltage dividers; **(2)** we have bought one voltage divider for our laboratory for measurement purposes?

2.4. A 1111 Ω resistance is composed by connecting a 1000, a 100, a 10 and a 1 Ω resistor in series. The tolerances (random relative errors) of the resistors are 0.01%, 0.1%, 1% and 10%, respectively. What is the tolerance of the 1111 Ω resistance, using the probabilistic summation of error components?

2.5. We construct a 900 Ω resistance using a 1 kΩ, a 10 kΩ, a 100 kΩ and a

1 MΩ resistor, which are connected in parallel. The tolerances (random relative errors) of the resistors are 0.01%, 0.1%, 1% and 10%, respectively. What is the worst case error of the 900 Ω resistance?

2.6. We are measuring the flow of water through a weir. The liquid flows through a V-shaped opening. The expression of the volume velocity is the following:

$$Q = \frac{4}{15}\sqrt{2g}\,\frac{d}{l}\,s^{5/2},$$

where d is the width of the weir, l is the height, s is the level of the liquid from the bottom of the weir, and g is the acceleration due to gravity. What is the most probable value of the measurement error, if the relative errors of d and l are 1%, and the relative error of s is 3%?

2.7. Our task is to measure small distances in a mechanical system. For this purpose metal sheets are fixed on the elements we want to measure. This results in a capacitor which can be used as a part of an RC oscillator. The distance is calculated from the frequency of this oscillator. The required formulas are: $C = \varepsilon A/d$, $f = 1/(2\pi RC)$; $\varepsilon = 8.85 \cdot 10^{-12}$ F/m, $A = 50$ cm^2, $R = 10$ kΩ. The error sources are the uncertainty in the frequency measurement and the uncertainty of the resistor value (1% relative error for each), the other error sources are neglected.

a) What is the worst case relative error of the distance measurement?

b) During the test of the equipment it turns out that the capacitance of the wires of the capacitor cannot be neglected, which is connected in parallel with the capacitor. What is the error of the measurement if the capacitance of the wires is $C_p = 45$ pF, and the nominal value of the distance is $d = 1$ mm?

2.8. The velocity of a liquid in a pipe is measured with ultrasound. We place two acoustic transceivers at the opposite sides of the pipe. The line connecting them has an α angle with the cross-section of the pipe. The travel time of the sound differs in the two directions due to the flow, and the velocity of the liquid can be calculated using the following formula:

$$v = \frac{l}{2\sin\alpha}\left[\frac{1}{t_1} - \frac{1}{t_2}\right],$$

where l is the distance between the transceivers, t_1 and t_2 are the propagation times. In our case $l = 0.5$ m, and the nominal velocity of the flow is $v = 5$ m/s, the propagation speed of the sound in the liquid is $c = 1500$ m/s, and $\alpha = 30°$.

a) Determine the required accuracy (relative error) of the propagation time measurements if the allowed maximal error of the velocity measurement is 5%!

b) What is the relative error of the velocity measurement if the propagation time measurements have $h_{\mathrm{syst}} = 1\%$ systematic and $h_{\mathrm{random}} = 50$ ppm random error?

2.9. We are measuring the height of a building based on the air pressure difference between the ground floor and the top of the building. The height can be calculated using the so-called barometric formula:

$$p(l) = p_0 e^{-\frac{\rho_0 g l}{p_0}},$$

where p is the pressure, $p_0 = 10^5$ Pa is the pressure at sea-level, $\rho_0 = 1.29\ \mathrm{kg/m^3}$ is the sea-level air density, $g = 9.81\ \mathrm{m/s^2}$ is the acceleration due to gravity, l is the height above sea-level.

a) Calculate the height of the building if the pressure at the ground floor is $p_1 = 99$ kPa, and the pressure at the top of the building is $p_2 = 98$ kPa !

b) The measurements are carried out in two ways. First we do a measurement in parallel with two different instruments at the top and the ground floor of the building, then we use only one instrument for pressure measurement, first at the ground floor and then at the top. What is the relative error of the height measurement if the maximum offset error of the instruments is $p_{\mathrm{off}} = 200$ Pa, and the maximum scaling error of the reading is $\varepsilon = 0.1\%$. For a given barometer, the offset and scaling errors can be considered constant (can be considered as systematic errors), but the errors can be different across the two barometers.

Chapter 3

Error calculation II.

3.1. ξ is a uniformly distributed random variable in the interval $[-1, 1]$. Draw its probability density function, determine its expected value and standard deviation!

3.2. A measured quantity can be modeled with a random variable. The probability density function has a constant value in the $[1, 2]$ and in the $[3, 4]$ intervals, otherwise it is 0. Determine the expected value and standard deviation of the random variable! Determine the width of the interval in which the measured values can be found with 90% probability! What are the bounds of this interval?

3.3. x is a normally distributed random variable and its value is between 1 and 2 with 99.7% probability. Estimate the standard deviation of x!

3.4. Smurf village wants to be member of the Smurf Union. For this purpose they have to standardize their main export item, the canned blueberries. Since Handy constructed a blueberry counting device, every can contains exactly 120 blueberries. The weight of one blueberry is within 4.5 g and 5.5 g with uniform distribution. Determine the 98% confidence interval for the weight of one can!

3.5. We are generating normally distributed samples with computer. A software generates uniformly distributed random variables in the $[0, 1]$ interval. A normally distributed sample is created by summing 48 uniformly distributed samples. Specify the required further operations to transform this variable so that it has standard normal distribution!

3.6. We are measuring a constant value which is disturbed by independent Gaussian noise with zero mean. The following six measurements were made:

$$13.6720 \quad 9.4190 \quad 21.3489 \quad 9.7298 \quad 14.6773 \quad 18.5959.$$

Determine the 90% confidence interval for the constant value!

3.7. The lengths of 3 tables are measured. The results are 100 ± 1 cm, 135 ± 1 cm and 65 ± 0.5 cm. The measurements are unbiased, normally distributed with a 95.5% confidence level. What is the maximal total length of the 3 tables with a 99.7% confidence level if they are placed one after the other? In other words, what is the space where the three tables fit with 99.7% probability?

3.8. Velocity is measured by the measurement of time and displacement. The measured value for the displacement is $x = 2000$ m $\pm$ 0.5%, and for the time it is $t = 2000$ s $\pm$ 0.1%. The distribution of the measurement errors is Gaussian, the confidence level is 90%. Determine the extended uncertainty of the velocity, if the extension factor is $k = 2$!

3.9. The value of a resistor is obtained by the measurement of its voltage and current. Two different instruments are used in the measurement. Determine the resistance and its standard uncertainty if the measured voltage is 1 V, its standard deviation is 0.01 V, the measured current is 1 mA and its standard deviation is 10 μA!

3.10. We are generating standard normally distributed samples. A software is used for this purpose which returns the values a or $-a$ with 50-50% chance, and $a = 2$. A normally distributed sample is created by generating $N = 256$ samples and summing them. Give the required operations to transform this random variable so that it has standard normal distribution!

3.11. The weight of a specific coin is measured. The standard deviation of the weight of various items from the same type of coin is assumed to be negligible compared to the precision of the weight-measurement. The systematic error of the measurements is zero, the random error is normally distributed with zero mean. The measurements are done using $N = 20$ laboratory scales of the same type. The estimate of the weight of the coin is calculated as the average of the 20 measurements. Determine the 99% confidence interval for the weight of the coin in the following two cases:

a) the same coin is measured with each instrument, the mean of the measurements is $m_1 = 3$ g and the estimated standard deviation is $s_1 = 0.02$ g,

b) a pack of $K = 40$ coins is measured with each instrument, the mean of the measurements is $m_K = 120$ g and the estimated standard deviation is $s_K = 0.02$ g!

3.12. The precision of clocks is tested in a clock factory. The daily systematic error of the clocks is assumed to be a constant value. The clocks also have a daily random error which is a normally distributed variable. The clock under test was set to 12.00.00 at noon, then in the next days at noon the displayed time was recorded and they have obtained

12.00.09 12.00.18 12.00.32 12.00.41 12.00.51 12.01.03 [h,min,sec].

Determine the 95% confidence interval for the daily systematic error of the clock!

3.13. We would like to determine the average height of the students studying measurement technology. The first measurements are

$$N_1 = 10;\ \bar{x} = 178 \text{ cm};\ s = 5.2 \text{ cm},$$

where N_1 is the number of measurements, $\bar{x}$ is the average of the measurements, and s is the empirical standard deviation.

a) Give the confidence interval for the average height of the students with $p = 90\%$!

b) How this confidence interval would change if the above mean and standard deviation values were calculated using $N_2 = 326$ measurements?

3.14. We measure current using a digital voltmeter and a normal resistor. The current is determined as the ratio of the voltage and the resistance. The value of the resistor is $R = 100.123 \pm 0.046\ \Omega$ at $T_0 = 20$ °C temperature. The measurement is done at $T = 26$ °C and the temperature coefficient of the resistance is $\alpha = 2 \cdot 10^{-5}$ 1/°C. The voltage was measured five times and the results are

138.75 mV   138.78 mV   138.72 mV   138.69 mV   138.74 mV.

a) Determine the estimate and the type-A standard uncertainty of the voltage!

b) Determine the type-B standard uncertainty of the voltage if the range of the voltmeter is 200 mV and the error of the measurement is $h = 0.02\%$ o.v. $+ 0.005\%$ o.r, where "o.v." stands for "of value" and "o.r." stands for "of range". The error components include the quantization error as well.

c) Determine the best estimate of the voltage and resistance, and their standard uncertainties!

d) Determine the value of the measured current and its standard uncertainty with $k = 2$ extension factor! The number of digits in the result should represent the accuracy of the result!

e) Estimate the probability of the confidence interval to which the above result corresponds!

f) The error h of the voltmeter was assumed to be a random error. Why do we consider this error to be random and how could this be shown experimentally?

Chapter 4

Measurement of voltage and current

4.1. A noisy sinewave is measured. Determine the RMS value of the sine wave if the measured RMS value (sine + noise) is $U_m = 6.1$ V, and the signal-to-noise ratio is SNR = 14.7 dB!

4.2. Determine the expected value (DC component), RMS value and fundamental frequency of the following signals:

a) $x(t) = A^2 \sin^2(2\pi f_0 t)$;

b) $x(t) = 1 \cdot \sin(3\pi f_0 t) - 0.9 \cdot \sin(3\pi f_0 t)$;

c) $x(t) = 12 \sin(2\pi f_0 t) + 12 \sin(6\pi f_0 t)$;

d) $x(t) = 12|\cos(2\pi f_0 t)|$;

e) $z(t) = \sqrt{2} e^{j2\pi f_0 t}$!

4.3. We are measuring a voltage having a nominal value of $U = 160$ V, but unfortunately the range of our voltmeters is limited to 100 V. Therefore, two moving-coil galvanometers of the same type are connected in series. Determine the worst case error of the measurement if the accuracy class of each instrument is 1 (that is, $h_{\text{o.r.}} = 1\%$)!

4.4. A digital voltmeter displays 0.0245 V in 2 V range. Estimate the error of the measurement if the data sheet or user's manual is not available!

4.5. The first five harmonic components of a periodic signal were measured. The RMS values of the components in dBV (0 dB corresponds to 1 V) are:

$$0 \quad -12 \quad -24 \quad -36 \quad -48 \quad [\text{dB}].$$

a) Determine the RMS value of each component in volts!

b) Determine the RMS value of the periodic signal!

c) Determine the total harmonic distortion (THD)!

4.6. A non-symmetric square wave is generated. One period of the signal is $T = 10$ ms long, within one period its value is $U_p = 5$ V for $T_1 = 4$ ms and 0 V for $T_2 = 6$ ms.

a) Determine the mean value, RMS value, crest factor and form factor of the signal!

b) The voltage is measured with an AC-coupled peak meter. What is the displayed value?

4.7. We measure a 1 V square wave with an absolute mean meter, a peak meter and a true RMS meter. What do these three voltmeters display?

4.8. We are measuring the input resistance of a circuit. The maximum allowed input voltage of the circuit is 100 mV. First a voltage generator is connected to the input and both the current and voltage are measured with two multimeters of the same kind.

a) Determine the input resistance and the relative error of the measurement if the measured voltage is $U_1 = 87.65$ mV in 200 mV range and the measured current is $I_1 = 01.72$ μA in 200 μA range. The error of both measurements is 0.05% of value and 0.002% of range.

The above measurement is not accurate enough, so a potentiometer is connected in series between the generator and the input. The value of the potentiometer is changed until the measured signal is $U_2 = U_1/2$ which means that the value of the input resistance equals with the resistance of the potentiometer.

b) Determine the relative error of the measurement of the input resistance if the voltage is measured with the same instrument and the tolerance of the potentiometer is 0.1% !

4.9. We are measuring a non-symmetric square wave having the period time $T = 500$ μs. The signal is 3 V for $\tau = 100$ μs, while for the rest of the period it is 0 V.

a) Determine the first 10 components of the Fourier series!

b) We measure this signal with a true RMS meter. What does the voltmeter measure and display?

c) What does the true RMS meter display if the signal is filtered with a $f_c = 5$ kHz lowpass filter? (The filter can be considered ideal, all components below f_c get through without any change, while it rejects everything above.)

Chapter 5

Measurement circuits

5.1. Two thermistors are used in a bridge circuit with 5 V DC voltage supply. The value of the two conventional resistors is 100 Ω. The resistance of each thermistor is 100 Ω at 20 °C temperature. Plot the wiring diagram! Determine the output voltage of the bridge if the resistance of the thermistors increase by $\Delta R = 1$ Ω due to the temperature change! Determine the output voltage and the systematic measurement error if each thermistor is connected to the bridge by wires both having 1 Ω resistance!

5.2. The division ratio of a compensated voltage divider is 1:10. The resistance of the lower component is 100 kΩ, and a 100 pF capacitor is connected in parallel. Determine the resistance and capacitance of the upper part of the divider!

5.3. The two inputs of a balanced analog multiplier are two sine waves with 10 V peak value, with the same phase and same frequency. The transfer factor of the multiplier is $k = 0.1$ 1/V ($u_{\text{out}}(t) = k\, u_{\text{in},1}(t)\, u_{\text{in},2}(t)$). Determine the mean value, absolute mean value and RMS value of the output signal of the multiplier!

5.4. Two thermistors are used for measuring the temperature with a measurement bridge.

a) How should we construct the bridge if the interval to be measured is $0 \ldots 50$ °C and the resistance of the thermistors is 100 Ω at 25 °C temperature?

b) Determine the supply current if the voltage measured across the thermistors should be 1 V at 25 °C!

c) Determine the output voltage at 40 °C if the temperature coefficient of the thermistor is $\alpha = 200$ ppm/°C?

d) Determine the required voltage gain of the output amplifier stage if the temperature interval to be measured should correspond to an output voltage interval of ± 10 V!

5.5. The effect of a force on an iron console is measured using two strain gauge resistors. One of the strain gauges elongates (its resistance increases), while the other compresses (its resistance decreases). The strain gauges are used in a measurement bridge which has two other ordinary resistors. The bridge has voltage supply.

a) Determine how the resistors should be placed in the bridge if the output voltage has to be a linear function of the resistance change!

b) If the mechanical system is unloaded (no stress), the output voltage of the bridge is 0 V. Determine the output voltage if the supply voltage is $U_S = 10$ V, the nominal value of the resistors is $R = 400\ \Omega$, and the relative change in the strain gauge resistances is 0.2%!

c) Determine the worst case measurement error if the tolerance of each strain gauge is 0.2%, and the tolerance of the conventional resistors is 0.5%!

5.6. The input signals of a balanced multiplier are two sine waves having the same frequency and 10 V peak value. The phase of the second sine wave is shifted by 90° compared to the first signal. The transfer factor of the multiplier is $k = 0.1$ 1/V ($u_{\text{out}}(t) = k\, u_{\text{in},1}(t)\, u_{\text{in},2}(t)$). Determine the mean value, absolute mean value and RMS value of the signal at the output of the multiplier!

5.7. The two inputs of a balanced multiplier are two sine waves. The first sine wave has a peak value of 10 V and a frequency of 50 Hz, while the second one has a peak value of 1 V and a frequency of 100 Hz. The transfer factor of the multiplier is $k = 0.1$ 1/V ($u_{\text{out}}(t) = k\, u_{\text{in},1}(t)\, u_{\text{in},2}(t)$). Determine the mean value and RMS value of the output of the multiplier!

5.8. An inverting amplifier is built which has a prescribed nominal gain of $A_0 = -5$. We are using $R_1 = 1$ kΩ and $R_2 = 5.1$ kΩ resistors in the circuit.

a) Plot the wiring diagram and determine the systematic error of the gain!

To reduce the systematic error, R_2 is connected in parallel with $R_3 = 270$ kΩ.

b) Determine the new systematic error of the gain!

The R_1 and R_2 are resistors have 0.1% accuracy, while R_3 is a resistor of 5% tolerance.

c) Determine the relative error of the gain using worst case summation of the error components!

5.9. A noninverting amplifier with a gain of $A_0 = 10$ is designed. We are using $R_1 = 1$ kΩ and $R_2 = 9.1$ kΩ resistors.

a) Plot the wiring diagram and determine the systematic error of the gain!

R_2 is substituted with the series connection of $R_3 = 6.8$ kΩ and $R_4 = 2.2$ kΩ to reduce the systematic error.

b) Determine the new systematic error of the gain!

All resistors have a tolerance of 0.1%.

c) Determine the relative error of the gain using worst case summation!

5.10. A differential amplifier with a prescribed symmetric gain of $A_{s,0} = 100$ is designed. The available parts are the following: an operational amplifier and resistors with values $R_1 = 1.5$ kΩ, $R_2 = 2.8$ kΩ, and $R_3 = 150$ kΩ, $R_4 = 280$ kΩ.

a) Plot the wiring diagram and indicate the resistors clearly!

b) Determine the common-mode rejection ratio in dB using worst case summation, if the tolerance of each resistor is 0.2%!

5.11. A symmetric triangular wave of 1 V peak value and 50 Hz frequency is measured with a moving coil voltmeter. This is done by connecting the output of an active one-way rectifier to the voltmeter. All the resistors used in the circuit are of $R = 1$ kΩ and have 1% tolerance. The opening voltage of the diode is $U_d = 0.6$ V, the full-scale range of the voltmeter is 1 V and its accuracy class is 0.5 (that is, $h_{\text{o.r.}} = 0.5\%$). The operational amplifier is assumed to be ideal.

a) Plot the wiring diagram and the waveform at the input of the voltmeter!

b) Determine the voltage displayed by the voltmeter!

c) Determine the relative error of the measurement using standard summation of the error components and $k = 2$ extension factor! Assume a uniform distribution of the errors of the resistors and the voltmeter.

5.12. An instrumentation amplifier is constructed. Besides the 3 operational amplifiers, 4 resistors of 25 kΩ, 2 of 5 kΩ and 1 piece of 5.55 kΩ are available.

a) Plot the wiring diagram and place the resistors to obtain a symmetric gain of 50!

b) Determine the relative systematic error of the symmetric gain!

c) Determine the minimal value (worst case) of the common mode rejection ratio if the tolerance of each resistor is 0.02%!

Chapter 6

Time and frequency measurement

6.1. Determine the value of the highest frequency which can be measured using a digital frequency meter if the measuring time is 10 ms and the largest number that can be represented by the counter is 10^5 !

6.2. Phase-shift is measured based on the Lissajous-plot. The oscilloscope displays an ellipse which has a vertical envelope of $a = 3$ cm and a vertical section of $b = 2.9$, the reading uncertainty is $h = 2\%$.

a) Determine the phase-shift if the major axis of the ellipse is **(1)** in the 1st and 3rd; **(2)** in the 2nd and 4th quarter!

b) Determine the absolute error of the phase measurement!

6.3. The frequency of a signal of a noiseless sine-wave generator is measured using a counter. The nominal frequency is $f_x = 100$ kHz and the clock frequency of the counter is $f_0 = 10$ MHz.

a) Determine the relative error of the time period measurement if only one period of the signal is measured!

b) The measurement error can be decreased by measuring more periods of the signal (average time period measurement). How many periods have to be measured to decrease the relative error under 10^{-4}?

c) If we want to decrease the relative error even more, the signal cannot be assumed noiseless. In this case, multiple measurements have to be averaged. Determine the number of measurements to be averaged to decrease the relative error until 10^{-5}!

d) How many measurements have to be averaged to decrease the relative error to the value of 10^{-4} if we can only measure single periods, as in question **a)**? Determine the distribution of the averaged and non-averaged measurement results!

6.4. The frequency of a periodic signal of 1325 Hz nominal value is measured using a counter-based period time meter. The measurement time is $t_{\mathrm{m}} = 0.1$

sec. Determine the relative error of the measurement if the clock frequency is 10 MHz and the clock has no error!

6.5. The clock frequency of a programmable frequency and period time meter is $f_0 = 10^7$ Hz with 10^{-6} relative error. A noiseless sine-wave of $f_x = 500$ kHz is measured.

a) Time period or frequency measurement should be used if the goal is the highest measurement accuracy during a given amount of t_m measurement time?

b) Determine the relative error of the measurement using worst case summation of the error components, if the measurement time is $t_m = 200$ μs!

c) Determine the error if the measurement time is $t'_m = 20$ ms! What kind of additional errors should be considered in this case?

6.6. We are using a two-input counter (A and B). The device measures period and frequency with a single input A, and time delay if both inputs are used. Even if the device displays frequency, period time is measured internally, and an arithmetic unit calculates the frequency. The clock frequency of the counter is $f_0 = 50$ MHz with $h = 3 \cdot 10^{-5}$ tolerance. A pure sine-wave of $f_x = 1.2$ kHz nominal frequency is measured which is connected to the input of a linear system and to input A of the instrument. The phase shift at the output of the system is $\varphi = 8°$ which should be measured as accurately as possible. The output is connected to input B and time interval is measured between inputs A and B. The phase shift is determined based on the frequency and time delay measurements, the measurement time is $t_\mathrm{m} = 0.1$ in both cases.

a) Determine the relative error of the frequency measurement!

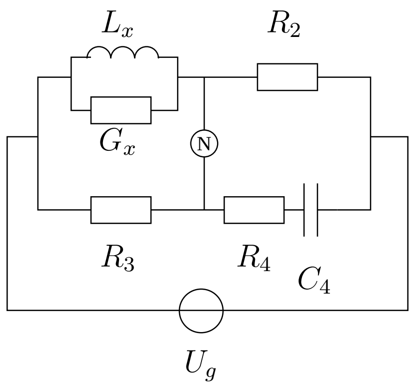
b) Determine the absolute error of the measured phase shift if the measurement of time delay is started by the rising edge of the signal at input A and stopped by the rising edge of the signal at input B! During the measurement time the time delay is measured in each period of the signal and these values are averaged by the arithmetic unit.

c) What do you think, can the precision of the measurement of phase shift be increased by triggering the time interval measurement with the falling edge of the signal on input A instead of the rising edge?

Chapter 7

Impedance and power measurement

7.1.



The so-called Hay-bridge in the above figure measures the equivalent parallel circuit model (L_x, G_x) of an inductance. The adjustable components are R_4 and C_4, while $R_2 = R_3 = 1$ kΩ.

a) Determine the condition of balance and the values of L_x and G_x if $R_4 = 100$ Ω and $C_4 = 100$ nF at $\omega = 1000$ 1/s!

b) At $\omega' = 2000$ 1/s the bridge is balanced for $R_4' = 25$ Ω and $C_4' = 100$ nF. Is the parallel RL circuit a good model of the inductance? If not, determine a more realistic model!

7.2. We measure the equivalent series RL circuit of an impedance. Determine the quality factor (Q), loss factor (tgδ) and dissipation factor (D) of the impedance! Determine the parallel RL, series RC and parallel RC equivalent circuits!

7.3. Plot the wiring diagram for connecting a resistor to an instrument which supports four-wire resistance measurement! Determine the voltage on each measuring lead! The resistor under test is $R_x = 1$ Ω and the resistance of each measuring lead is $R_s = 100$ mΩ (this contains the resistance of the plug as well), and the measuring current is $I = 100$ mA!

7.4. We measure an $R = 10\ \Omega$ resistor with four-wire measurement. The measuring frequency is 100 Hz, the resistance of each measuring lead is 0.1 Ω. Determine the worst-case measurement error of the resistance measurement if the tolerance of the current- and voltage measurement is 0.5%! The ammeter and voltmeter can be considered ideal ($R_v = \infty$ and $R_a = 0$).

7.5. We measure an $R = 10\ \Omega$ resistor with three-wire measurement. The measuring frequency is 100 Hz, the resistance of each measuring lead is 0.1 Ω. Determine the worst-case measurement error of the resistance measurement if the tolerance of the current- and voltage measurement is 0.5%! The ammeter and voltmeter can be considered ideal ($R_v = \infty$ and $R_a = 0$).

7.6. We measure an $R = 10\ \Omega$ resistor with five-wire measurement. The measuring frequency is 10 kHz, the resistance of each measuring lead is 0.1 Ω. Determine the worst-case measurement error of the resistance measurement if the tolerance of the current- and voltage measurement is 0.5%! The ammeter and voltmeter can be considered ideal ($R_v = \infty$ and $R_a = 0$).

7.7. We are constructing a model of a magnetic-core inductance. First we measure the equivalent series circuit with an impedance meter at 50 Hz frequency and we obtain $R_e = 0.5395\ \Omega$ and $L_e = 20$ mH. Next, the DC resistance of the coil is measured with an ohmmeter, and we read $R_s = 0.5\ \Omega$. Determine an appropriate three parameter model for the magnetic-core inductance which represents the inductance, the core losses, and the winding (or copper) losses!

7.8. We measure an $R_x = 100\ \Omega$ resistor of a circuit using a three-wire impedance meter. Both ends of the resistor are connected to the ground with $R_g = 1$ kΩ resistors. The voltmeter is ideal, (i.e., $R_v = \infty$), but the input resistance of the ammeter is $R_A = 1\ \Omega$.

a) Calculate the relative error of the measurement of R_x!

b) Determine the error caused solely by the R_g resistors, despite the three-wire measurement!

7.9. We measure power with the 3-voltmeter method. The supply voltage is $U_G = 10$ V and the value of the reference resistor is $R_R = 100\ \Omega$. The voltage drops on the reference resistor and on the impedance under test are both $U_N = U_Z = 5.8$ V.

a) Determine the dissipated active power and the value of $\cos\varphi$!

b) Determine the relative error of the measurement using $k = 2$ extension factor if the accuracy class of all voltmeters is 0.5 ($h_{\text{o.r.}} = 0.5\%$) and the measurement range is $U_{\max} = 10$ V. The errors are uniformly distributed and the error of the reference resistor can be neglected!

c) Determine whether the load is inductive or capacitive!

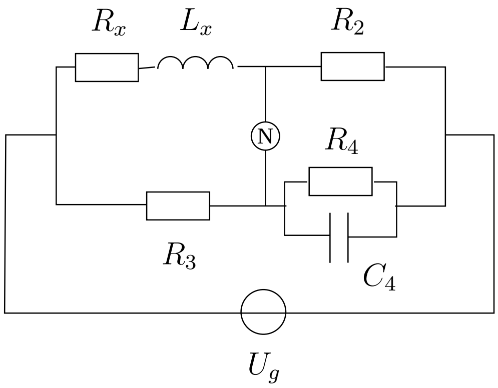
7.10. We measure impedance with the 3-voltmeter method. The supply voltage is $U_g = 10.000$ V, the value of the reference resistor is $R_R = 100\ \Omega$, and the voltage drops on the reference resistor and the impedance under test are $U_N = 07.053$ V and $U_x = 06.877$ V, respectively.

a) Determine the absolute value and phase of the impedance!

b) The uncertainty of the voltmeter is unknown, but its display is digital. In 20 V range the voltmeters display exactly the same numbers as given above, that is, $U_g = 10.000$ V, $U_N = 07.053$ V, and $U_x = 06.877$ V. The uncertainty of the reference resistance is 0.01%. Determine the worst case relative error of the absolute value of the impedance based on the available information!

c) Determine whether the absolute value or the phase of the impedance can be measured more precisely!

7.11.



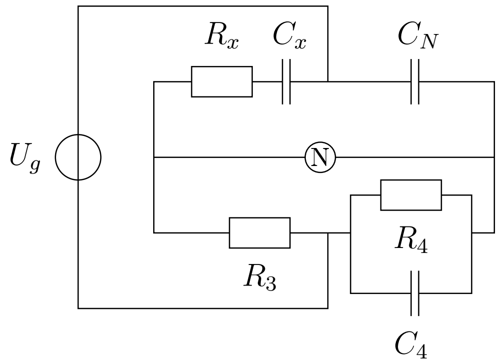
The so-called Maxwell–Wien bridge in the above figure measures the equivalent series model (L_x, R_x) of an inductance. The adjustable elements of the bridge are R_4 and C_4, and $R_2 = R_3 = 100\ \Omega$.

a) Determine the condition of balance and the value of L_x and R_x, if at $f = 159.1$ Hz frequency $R_4 = 10$ kΩ and $C_4 = 500$ nF!

b) Determine the quality factor of the inductance!

c) Determine the measurement error of R_x if the loss factor of C_4 is $D_4 = 0.002$ at the measurement frequency of $f = 159.1$ Hz!

7.12.



The so-called Schering bridge in the above figure measures the equivalent series model (C_x, R_x) of a capacitor. The adjustable elements are R_3 and C_4, and $R_4 = 10$ kΩ, $C_N = 10$ nF.

a) Determine the condition of balance and the value of C_x and R_x, if at $\omega = 1000$ 1/s $R_3 = 909\ \Omega$ and $C_4 = 1.11$ nF!

b) Determine the loss factor (tgδ) of the capacitor!

c) How can this bridge be used to perform an insulation test?

7.13. We measure the capacitance of a capacitor mounted in a metal box. The nominal value of the capacitor is 2 nF.

a) Between each wire of the capacitor and the metal box there is a 100 pF stray capacitance. Determine the relative error of the measurement due to the stray capacitances, when the two wire measurement method is used!

b) What kind of measurement layout should be used if we wish to cancel the above error?

c) How could we measure the value of the stray capacitances?

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Chapter 8

AD- and DA-converters

8.1. The reference voltage of a $b = 12$ bit ADC is $U_r = 1$ V with $h_r = \pm 0.05\%$ error. We measure a DC voltage with nominal value $U_x = 0.15$ V by the ADC. Determine the relative and absolute errors of the measurement in the worst case!

8.2. In Switzerland alternate voltage of 16 2/3 Hz is used for heavy rail traction. One of the instruments of an electric locomotive contains a dual-slope ADC. How should we choose the integration time so that the error caused by the traction current is eliminated? Do we need to change the integration time if we want to use the instrument in Hungary, where the network frequency is 50 Hz?

8.3. A dual slope ADC converts voltage in the $[0, 1]$ V range. The reference voltage is $U_r = 1$ V.

a) Determine the maximum allowed relative error of the reference voltage if the ADC has 20 bits resolution! The error of time measurement can be neglected.

b) Determine the integration time if the effect of sinusoidal disturbances with frequencies 50 Hz and 60 Hz should be suppressed!

c) Determine the maximum allowed error of the time measurement to achieve 20 bits resolution if the error of the integration time can be neglected!

8.4. A dual-slope ADC has a reference voltage $U_r = 2$ V with $h_r = \pm 80$ ppm tolerance. Inside the converter the time is measured by a $f_0 = 20$ MHz crystal oscillator. The error of the oscillator is negligible. The converter first integrates the input signal for $T = 20$ ms, which is an integer multiple of the clock cycle provided by the crystal oscillator. Then the reference voltage is integrated and the time is measured by counting the full periods of the clock cycle. The measured voltage is then calculated by an arithmetic unit.

a) Determine the *resolution* of the ADC in bits!

b) Determine the *accuracy* (useful number of bits) of the converter!

Part II

Solutions

Chapter 1

Basic problems

1.1. The distance taken by the car is the time-integral of the velocity function $v(t)$, written as

$$s = \int_0^T k\ f(t)dt,$$

where k is the coefficient which synchronizes the dimensions. Considering that $f(t)$ is a semicircle, we can write

$$f\left(\frac{T}{2}\right) = \frac{T}{2}.$$

According to the figure

$$k\ f\left(\frac{T}{2}\right) = v_{\max},$$

therefore

$$k = \frac{2v_{\max}}{T}.$$

Because $f(t)$ is a semicircle, the taken distance is

$$s = \frac{k}{2}\frac{T^2}{4}\pi = \frac{v_{\max}T\pi}{4} = 261.8\ \text{m}.$$

1.2. Although the below expressions are given for ergodic signals, the validity of the units is general. Note that in terms of dimensions, integration means multiplication with the integration variable.

a)

$$P = \lim_{T\to\infty}\frac{1}{T}\int_0^T x^2(t)dt\ \Rightarrow\ [P] = 1\ \text{V}^2;$$

b)

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\ \Rightarrow\ [X(f)] = 1\ \text{Vs} = 1\ \text{V/Hz};$$

c)

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt \;\Rightarrow\; [R(\tau)] = 1\ \text{V}^2;$$

d)

$$S(f) = \mathcal{F}\{R(\tau)\} \;\Rightarrow\; [S(f)] = 1\ \text{V}^2\text{s} = 1\ \text{V}^2/\text{Hz};$$

e)

$$E(f) = |X(f)|^2 \;\Rightarrow\; [E(f)] = 1\ \text{V}^2\text{s}^2 = 1\ \text{V}^2\text{s}/\text{Hz};$$

f)

$$X_{\text{eff}} = \sqrt{P} \;\Rightarrow\; [X_{\text{eff}}] = 1\ \text{V};$$

g)

$$X_{\text{RMS}} = \sqrt{P} \;\Rightarrow\; [X_{\text{RMS}}] = 1\ \text{V};$$

h)

$$\sigma^2 = E\{(x-\mu)^2\} = E\{x^2\} - \mu^2 = P - \mu^2 \;\Rightarrow\; [\sigma^2] = 1\ \text{V}^2;$$

i)

$$\Psi = E\{x^2\} = P \;\Rightarrow\; [\Psi] = 1\ \text{V}^2;$$

j)

$$\mu = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)dt \;\Rightarrow\; [\mu] = 1\ \text{V};$$

k) based on the solution of **h)**:

$$[\sigma] = 1\ \text{V}.$$

1.3. The time domain expression of the output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Then we write the above expression in terms of the units of the variables:

$$[y] = [h][x][t].$$

Which means that the unit of the impulse response is

$$[h] = \frac{[y]}{[x][t]} = 1\ \frac{\text{A}}{\text{Vs}} = 1\ S\text{Hz} = 1\ \frac{S}{s}.$$

1.4. The complex trigonometric expression of the signal is

$$\begin{aligned} x(t) &= A\cos(2\pi f t + \varphi) = \frac{A}{2}\left(e^{j(2\pi f t+\varphi)} + e^{-j(2\pi f t+\varphi)}\right) = \\ &= \frac{A}{2}e^{j\varphi}e^{j2\pi f t} + \frac{A}{2}e^{-j\varphi}e^{-j2\pi f t}. \end{aligned}$$

The general form of the complex Fourier series is

$$x(t) = \sum_{-\infty}^{\infty} C_n e^{j2\pi f_0 t n},$$

where f_0 is the fundamental frequency of the signal (the time period is $T = 1/f_0$). Since we have only one frequency component in $x(t) = A\cos(2\pi f t + \varphi)$, that will be the fundamental, that is, $f_0 = f$.

We have to set C_n so that the complex Fourier series equals the expression of $x(t)$. We see that we need two complex exponentails, meaning that only the C_1 and C_{-1} values will be nonzero, since $f_0 = f$:

$$C_n = \begin{cases} A/2\, e^{j\varphi}, & \text{if } n = 1 \\ A/2\, e^{-j\varphi}, & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}.$$

1.5. The fundamental frequency is the largest common divider of the frequencies of the components. The components have the frequency of f_a and $2.5f_a$. (We have to divide the angular frequencies given in the cosine functions by 2π to obtain the frequencies, that is $2\pi f_a/(2\pi)$ and $5\pi f_a/(2\pi)$.) Their largest common divider is $f_0 = 0.5f_a$, and the time period is $T = 1/f_0 = 2/f_a$.

By computing the complex form of $x(t)$ using Euler's formula similarly to the problem 1.5 (here $\varphi = 0$), we obtain

$$x(t) = \frac{A_1}{2}e^{j2\pi f_a t} + \frac{A_1}{2}e^{-j2\pi f_a t} + \frac{A_2}{2}e^{j5\pi f_a t} + \frac{A_2}{2}e^{j5\pi f_a t}.$$

which we again make equal to the general form of the complex Fourier series

$$x(t) = \sum_{-\infty}^{\infty} C_n e^{j2\pi f_0 t n}.$$

The component with frequency f_a is the second harmonic ($n = 2$) of the Fourier series since $f_a = 2f_0$, and the component having the frequency $2.5f_a$ is fifth ($n = 5$), since $2.5f_a = 5f_0$. Thus, the coefficients of the complex Fourier series are:

$$C_n = \begin{cases} A_1/2, & \text{if } n = 2, -2 \\ A_2/2, & \text{if } n = 5, -5 \\ 0 & \text{otherwise} \end{cases}.$$

1.6. By using trigonometric identities we get

$$x(t) = \frac{A_1 A_2}{2} \left\{ \cos[2\pi(f_2 - f_1)t] - \cos[2\pi(f_2 + f_1)t] \right\},$$

Therefore the frequencies of the two components are $f_a = 0.6f_1$ and $f_b = 2.6f_1$. The fundamental frequency is the largest common divider of the frequency components of the signal, which is $f_0 = 0.2f_1$ in this case. The period time is $T = 1/f_0 = 5/f_1$.

Since we were able to find the fundamental frequency, the signal is periodic. This is always the case if the frequencies have a rational ratio. On the other hand, for example, a signal having the components with frequencies f_1 and πf_1 is not periodic.

Chapter 2

Error calculation I.

2.1.

$$\begin{aligned} x &= 2000 \text{ m} \pm 5\% = (2000 \pm 10) \text{ m} = \hat{x} \pm \Delta x \rightarrow \hat{x} = 2000 \text{ m},\ \Delta x = 10 \text{ m}, \\ t &= 2000 \text{ s} \pm 0.1\% = (2000 \pm 2) \text{ s} = \hat{t} \pm \Delta t \rightarrow \hat{t} = 2000 \text{ s},\ \Delta t = 2 \text{ s}. \end{aligned}$$

The estimate of the velocity is

$$\hat{v} = \frac{\hat{x}}{\hat{t}} = 1 \ \frac{\text{m}}{\text{s}},$$

and the worst case error is

$$\Delta v \cong \left|\frac{\partial v}{\partial x}\Delta x\right| + \left|\frac{\partial v}{\partial t}\Delta t\right| = \left|\frac{1}{\hat{t}}\Delta x\right| + \left|-\frac{\hat{x}}{\hat{t}^2}\Delta t\right| = 6 \cdot 10^{-3} \ \frac{\text{m}}{\text{s}}. \tag{2.1}$$

So the result for the velocity can be written as

$$v = (1 \pm 6 \cdot 10^{-3}) \ \frac{\text{m}}{\text{s}} = 1 \ \frac{\text{m}}{\text{s}} \pm 0.6\%.$$

The relative value of the error can be calculated directly by dividing (1) with $\hat{v} = \hat{x}/\hat{t}$:

$$\frac{\Delta v}{\hat{v}} \cong \left|\frac{\Delta x}{\hat{x}}\right| + \left|-\frac{\Delta t}{\hat{t}}\right| = (0.5 + 0.1)\% = 0.6\%.$$

Remark. From this point the approximation signal $\cong$ will be replaced with the equal sign. In addition, the hat sign ˆ will be left behind also. These signs will be used only in the case when the indistinguishability of the actual and estimated values would be confusing.

2.2. The nominal value and the error of each resistor is

$$R_i = R_{\text{nom}} = 1 \text{ k}\Omega, \quad \Delta R_i = \pm\Delta R_{\text{nom}} = \pm h R_{\text{nom}} = \pm 10 \ \Omega,$$

where h is the tolerance (relative error) of the resistors. The total resistance is

$$R_e = \sum_{i=1}^{100} R_i = 100 R_{\text{nom}} = 100 \text{ k}\Omega.$$

We can solve the problem in two ways: we can either sum the absolute errors and then convert to relative error (I), or we can sum the relative error components directly (II).

I. The change of the total resistance for the change in the ith resistor's value is

$$\Delta R_e|_i = \frac{\partial R_e}{\partial R_i}\Delta R_i = \Delta R_i.$$

The total error for case **(a)** is

$$\Delta R_e = \sum_{i=1}^{100} |\Delta R_e|_i = 100\Delta R_{\text{nom}}, \quad \frac{\Delta R_e}{R_e} = \frac{100\Delta R_{\text{nom}}}{100 R_{\text{nom}}} = \frac{\Delta R_{\text{nom}}}{R_{\text{nom}}} = h = 1\%.$$

The total error for case **(b)** is

$$\begin{aligned}\Delta R_e &= \sqrt{\sum_{i=1}^{100} (\Delta R_e)_i^2} = \sqrt{100\Delta R_{\text{nom}}^2} = 10\Delta R_{\text{nom}},\\ \frac{\Delta R_e}{R_e} &= \frac{10\Delta R_{\text{nom}}}{100 R_{\text{nom}}} = 0.1\frac{\Delta R_{\text{nom}}}{R_{\text{nom}}} = 0.1h = 0.1\%.\end{aligned}$$

II. The relative change of the total resistance caused by the relative change of the ith resistor's value is

$$\left.\frac{\Delta R_e}{R_e}\right|_i = \frac{\partial R_e}{\partial R_i}\frac{R_i}{R_e}\frac{\Delta R_i}{R_i} = \frac{R_i}{R_e}h = \frac{R_{\text{nom}}}{100 R_{\text{nom}}}h = \frac{1}{100}h.$$

The total error for case **(a)** is

$$\frac{\Delta R_e}{R_e} = \sum_{i=1}^{100} \left|\frac{\Delta R_e}{R_e}\right|_i = h = 1\%.$$

The total error for case **(b)** is:

$$\frac{\Delta R_e}{R_e} = \sqrt{\sum_{i=1}^{100} \left(\frac{\Delta R_e}{R_e}\right)_i^2} = \sqrt{100 \cdot 10^{-4} h^2} = 0.1h = 0.1\%.$$

The two ways (I) and (II) lead to the same result as expected.

2.3.

a) The voltage division ratio is

$$a = \frac{R_2}{R_1 + R_2} = \frac{1}{50} = 0.02.$$

b) The error can be determined by first differentiating the expression above to obtain the sensitivities

$$c_{R_1} = \frac{\partial a}{\partial R_1} = \frac{-R_2}{(R_1 + R_2)^2}, \qquad c_{R_2} = \frac{\partial a}{\partial R_2} = \frac{R_1}{(R_1 + R_2)^2}.$$

The error components of the division ratio a coming from the errors of R_1 and R_2 are

$$\left.\frac{\Delta a}{a}\right|_{R_1} = \frac{c_{R_1} R_1}{a} \frac{\Delta R_1}{R_1} = \frac{-R_2}{(R_1+R_2)^2} \frac{R_1+R_2}{R_2} R_1 \frac{\Delta R_1}{R_1} = \frac{-R_1}{R_1+R_2} \frac{\Delta R_1}{R_1},$$

and

$$\left.\frac{\Delta a}{a}\right|_{R_2} = \frac{c_{R_2} R_2}{a} \frac{\Delta R_2}{R_2} = \frac{R_1}{(R_1+R_2)^2} \frac{R_1+R_2}{R_2} R_2 \frac{\Delta R_2}{R_2} = \frac{R_1}{R_1+R_2} \frac{\Delta R_2}{R_2}.$$

Since the worst case error has to be computed, the above components are summed with absolute value:

$$\frac{\Delta a}{a} = \left|\left.\frac{\Delta a}{a}\right|_{R_1}\right| + \left|\left.\frac{\Delta a}{a}\right|_{R_2}\right| = \frac{R_1}{R_1+R_2}\left(\frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2}\right) = 196 \text{ ppm}.$$

c) **1)** For the manufacturer of the voltage divider, this is a random error. The tolerance means that the error in each voltage divider may have any value within known bounds. So the error of the ratio will be random and it is within the bounds $[-196 \text{ ppm}, 196 \text{ ppm}]$ calculated above using the worst case summation.

2) For the user this is a systematic error. Our divider is made of resistances with fixed values, so the deviation from the nominal value of the ratio is always the same and can be taken into account during measurements (that is, the voltage divider can be *calibrated*).

2.4. The solution is exactly the same as that of problem 2.2, where we had 100 resistors instead of 4. Here we solve the problem using absolute errors. The absolute errors of the resistors are

$$\Delta R_i = h_i R_i$$

which are 0.1 Ω for all the resistors. The net resistance is

$$R_e = R_1 + R_2 + R_3 + R_4.$$

The sensitivities are

$$c_i = \frac{\partial R_e}{\partial R_i} = 1,$$

thus, the total absolute error of R_e with probabilistic summation is

$$\Delta R_e = \sqrt{\Delta R_1^2 + \Delta R_2^2 + \Delta R_3^2 + \Delta R_4^2} = 0.2\ \Omega,$$

which is then converted to relative error:

$$\frac{\Delta R_e}{R_e} = 0.018\%.$$

2.5. The inaccuracy of the total resistance can be traced back to the errors in each value of the used resistors, and in addition, to a systematic error, since the total resistance of the parallel connection is:

$$R_e = \frac{1}{\sum_{i=1}^{4} \frac{1}{R_i}} = 900.09\ \Omega$$

instead of the required $R_{\text{nom}} = 900\ \Omega$ nominal value, so the resistance has a

$$h_{\text{syst}} = 0.01\%$$

systematic error. The random error is computed by first taking the derivative of R_e to obtain the sensitivities:

$$c_i = \frac{\partial R_e}{\partial R_i} = -\frac{1}{\left(\frac{1}{\sum_{j=1}^{4} \frac{1}{R_j}}\right)^2} \left(-\frac{1}{R_i^2}\right) = \frac{R_e^2}{R_i^2}$$

The relative error components are

$$\left.\frac{\Delta R_e}{R_e}\right|_{R_i} = c_i \frac{R_i}{R_e} \frac{\Delta R_i}{R_i} = \frac{R_e}{R_i} \frac{\Delta R_i}{R_i}$$

and thus the worst case relative random error is

$$h_{\text{rand}} = \sum_{i=1}^{4} \left| \left.\frac{\Delta R_e}{R_e}\right|_{R_i} \right| = \sum_{i=1}^{4} \left| \frac{R_e}{R_i} \frac{\Delta R_i}{R_i} \right| = 0.036\%.$$

To determine the worst case total error, the random and systematic errors have to be summed:

$$\frac{\Delta R_e}{R_e} = h_{\text{total}} = h_{\text{syst}} \pm h_{\text{rand}} = 0.01\% \pm 0.036\%.$$

Note that since we know the sign of the systematic error, this leads to an asymmetric error interval

$$h_{\text{total}} = [-0.026\%, 0.046\%].$$

If we want to express the error with a single number, we choose the bound with the larger absolute value, that is

$$|h_{\text{total}}| \leq 0.046\%.$$

2.6. First we rewrite the formula of the measured volume velocity as

$$Q = \frac{4}{15}\sqrt{2g}\frac{d}{l}s^{5/2} = Kd^1 l^{-1} s^{5/2}.$$

Since this is a product of variables at different powers, the resulting relative errors can be simply computed by scaling the input errors with the corresponding powers:

$$\left.\frac{\Delta Q}{Q}\right|_{d} = 1\frac{\Delta d}{d},$$

$$\left.\frac{\Delta Q}{Q}\right|_l = -1\frac{\Delta l}{l},$$

$$\left.\frac{\Delta Q}{Q}\right|_s = \frac{5}{2}\frac{\Delta s}{s}.$$

To determine the most probable value of the error, the components have to be summed quadratically:

$$\frac{\Delta Q}{Q} = \sqrt{\left(\frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta l}{l}\right)^2 + \left(\frac{5}{2}\frac{\Delta s}{s}\right)^2} = 7.63\%.$$

2.7. Using the given expressions we obtain

$$d = \frac{\varepsilon A}{C} = 2\pi R f \varepsilon A = K f^1 R^1. \tag{2.2}$$

a) The worst case error can be computed simply noticing that Eq. (2.2) is the product of variables at the first power, thus we have

$$\frac{\Delta d}{d} = \left|\frac{\Delta f}{f}\right| + \left|\frac{\Delta R}{R}\right| = 2\%.$$

b) The capacitance C_p which connects in parallel with C is added to the value of the capacitor, thus, we meausre $C_m = C + C_p$. This causes a systematic error, which has a known value so it can be compensated:

$$C = C_m - C_p = \frac{1}{2\pi f R} - C_p.$$

Accordingly, Eq. (2.2) changes in the following way:

$$d = \frac{\varepsilon A}{C} = \frac{2\pi R f \varepsilon A}{1 - C_p 2\pi R f}. \tag{2.3}$$

Since the function has changed, the effects of the frequency measurement error and of the resistor uncertainty have to be recalculated. First the effect of the frequency error is evaluated (the details of the derivation are omitted here):

$$\left.\frac{\Delta d}{d}\right|_f = \frac{1}{1 - C_p 2\pi R f}\frac{\Delta f}{f}.$$

Using the above expression the actual value of the error can be determined. However, the expression contains an intermediate parameter (the frequency) which is the function of the variable we want to measure. The value of f can be expressed from (2.3), so the expression of the error can be simplified as:

$$\frac{\Delta d_f}{d} = \frac{\varepsilon A + C_p d}{\varepsilon A}\frac{\Delta f}{f} = \frac{C + C_p}{C}\frac{\Delta f}{f}, \tag{2.4}$$

It can be seen that when C is small compared to C_p (d is large), the error is much larger than for case **a)**, where it was simply $\Delta f/f$.

Since R in Eq. (2.3) is in the same position as f, the derivations for R would result in an expression having the same form as Eq. (2.4). As a result, $\Delta R/R$ has to be multiplied with the same parameter in the total error, so the worst case error is:

$$\frac{\Delta d}{d} = \frac{C + C_p}{C}\left[\frac{\Delta f}{f} + \frac{\Delta R}{R}\right] = 4.03\%.$$

2.8. The sensitivities are

$$c_1 = \frac{\partial v}{\partial t_1} = \frac{l}{2\sin\alpha}\left(-\frac{1}{t_1^2}\right)$$

$$c_2 = \frac{\partial v}{\partial t_1} = \frac{l}{2\sin\alpha}\left(\frac{1}{t_2^2}\right),$$

and thus the error components become

$$\left.\frac{\Delta v}{v}\right|_{t_1} = c_1\frac{t_1}{v}\frac{\Delta t_1}{t_1} = -\frac{t_2}{t_2 - t_1}\frac{\Delta t_1}{t_1}$$

$$\left.\frac{\Delta v}{v}\right|_{t_2} = c_2\frac{t_2}{v}\frac{\Delta t_2}{t_2} = \frac{t_1}{t_2 - t_1}\frac{\Delta t_2}{t_2}.$$

a) The error of the velocity measurement in the worst case is

$$\frac{\Delta v}{v} = \left|\frac{t_2}{t_2 - t_1}\frac{\Delta t_1}{t_1}\right| + \left|\frac{t_1}{t_2 - t_1}\frac{\Delta t_2}{t_2}\right|.$$

Since $t_1 \approx t_2$, and we can assume that the two time measurements have the same accuracy $\Delta t_1/t_1 = \Delta t_2/t_2 = \Delta t/t$, we can write

$$\frac{\Delta v}{v} = 2\left|\frac{t_2}{t_2 - t_1}\right|\frac{\Delta t}{t}. \tag{2.5}$$

By rearranging the original expression of v we obtain

$$\frac{t_2 - t_1}{t_2 t_1} = \frac{v 2\sin\alpha}{l},$$

where we substitute $t_1 \cong l/c$ in the numerator:

$$\frac{t_2 - t_1}{t_2(l/c)} \cong \frac{v 2\sin\alpha}{l},$$

giving

$$\frac{t_2 - t_1}{t_2} \cong \frac{v}{c}2\sin\alpha \cong \frac{1}{300}.$$

Now we can get back to the expression of velocity error Eq. (2.5):

$$\frac{\Delta v}{v} = 2\left|\frac{t_2}{t_2 - t_1}\right|\frac{\Delta t}{t} = 600\frac{\Delta t}{t}.$$

Solving this for $\Delta t/t$ gives

$$\frac{\Delta t}{t} = \frac{1}{600}\frac{\Delta v}{v} = 8.33 \cdot 10^{-5}.$$

b) For the systematic error, the error components already computed in **b)** have to be summed with sign (not by absolute value):

$$\left.\frac{\Delta v}{v}\right|_{\text{syst}} = -\frac{t_2}{t_2 - t_1}\left.\frac{\Delta t_1}{t_1}\right|_{\text{syst}} + \frac{t_1}{t_2 - t_1}\left.\frac{\Delta t_2}{t_2}\right|_{\text{syst}} = \frac{t_1 - t_2}{t_2 - t_1}h_{\text{syst}} = -h_{\text{syst}} = 1\%.$$

For the random error we have already developed the expression Eq. (2.5), now with a given $h_{\text{rand}} = 50$ ppm error in time measurement we get

$$\left.\frac{\Delta v}{v}\right|_{\text{rand}} = 2\left|\frac{t_2}{t_2 - t_1}\right|h_{\text{rand}} = 3\%.$$

The total error is the sum of the systematic and random error components:

$$\frac{\Delta v}{v} = \left.\frac{\Delta v}{v}\right|_{\text{syst}} \pm \left|\left.\frac{\Delta v}{v}\right|_{\text{rand}}\right| = [-2\%,\ 4\%],$$

which is again an asymmetric error interval. To express the error with a single number, we may write

$$\left|\frac{\Delta v}{v}\right| \leq 4\%.$$

2.9.

a) The pressure at height l_1 is

$$p_1 = p(l_1) = p_0 e^{-\frac{\rho_0 g l_1}{p_0}},$$

which leads to

$$\frac{p_1}{p_0} = e^{-\frac{\rho_0 g l_1}{p_0}},$$

$$\ln p_1 - \ln p_0 = -\frac{\rho_0 g l_1}{p_0}.$$

Thus,

$$l_1 = \frac{p_0}{\rho_0 g}(\ln p_0 - \ln p_1)$$

and similarly for l_2

$$l_2 = \frac{p_0}{\rho_0 g}(\ln p_0 - \ln p_2)$$

So the height of the building is

$$l = l_2 - l_1 = \frac{p_0}{\rho_0 g}(\ln p_1 - \ln p_2) = 80.22 \text{ m}.$$

b) The sensitivities are

$$c_1 = \frac{\partial l}{\partial p_1} = \frac{p_0}{\rho_0 g}\frac{1}{p_1}$$

$$c_2 = \frac{\partial l}{\partial p_2} = \frac{p_0}{\rho_0 g}\left(-\frac{1}{p_2}\right).$$

The components of the error are thus

$$\left.\frac{\Delta l}{l}\right|_{p_1} = c_1\frac{p_1}{l}\frac{\Delta p_1}{p_1} = \frac{1}{\ln p_1 - \ln p_2}\frac{\Delta p_1}{p_1}$$

$$\left.\frac{\Delta l}{l}\right|_{p_2} = c_2\frac{p_2}{l}\frac{\Delta p_2}{p_2} = -\frac{1}{\ln p_1 - \ln p_2}\frac{\Delta p_2}{p_2}.$$

The error of the height measurement is:

$$\frac{\Delta l}{l} = \frac{1}{\ln p_1 - \ln p_2}\left(\frac{\Delta p_1}{p_1} ? \frac{\Delta p_2}{p_2}\right) = c\,(e_1 \; ? \; e_2),$$

where at the place of the question mark the summation has to be done with respect to the chosen method (e.g., worst case or systematic).

To evaluate the error components first we study the case when two independent instruments are used:

$$e_1 = \frac{p_{\text{off,I}}}{p_1} + \varepsilon_{\text{I}}, \quad e_2 = \frac{p_{\text{off,II}}}{p_2} + \varepsilon_{\text{II}}.$$

where $p_{\text{off,I}}$ and $p_{\text{off,II}}$ are the offset errors, ε_1 and ε_2 are the scaling errors using instrument I and II, respectively. Since the errors are independent (we use two different barometers), they have to be summed by the probabilistic or by the worst case method. We choose worst case summation since we have only a few (two) error components:

$$\frac{\Delta l}{l} = c\,(|e_1| + |e_2|) \approx 60\%.$$

On the other hand, if the same instrument is used in the two measurements:

$$e_1 = \frac{p_{\text{off,I}}}{p_1} + \varepsilon_{\text{I}}, \quad e_2 = \frac{p_{\text{off,I}}}{p_2} + \varepsilon_{\text{I}}.$$

Using index I refers to the use of the same instrument. The errors are not independent since we are using the same barometer. Thus, they have to be summed with their signs:

$$\frac{\Delta l}{l} = c\,|e_1 - e_2| = c\left|\frac{p_{\text{off,I}}}{p_1} + \varepsilon_{\text{I}} - \frac{p_{\text{off,I}}}{p_2} - \varepsilon_{\text{I}}\right| = 0.2\%.$$

The above results lead to the following consequences:

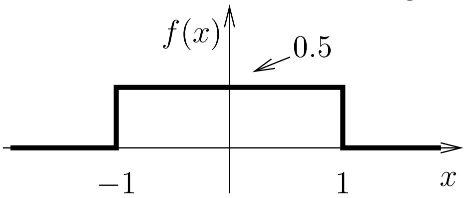
- Since the pressure difference is small, c has a large value, so even a small error in the measurement of the pressure results in a high error in the measurement of the height.
- When using the same instrument, the scaling errors ε_{I} cancel out completely, since the expression of the height contains the ratio of the pressures. On the other hand, the offset errors $p_{\mathrm{off,I}}$ of the two measurements do not completely cancel to zero, but still the effect of offset is significantly reduced. In summary, since the measurement errors are systematic, it is a better choice to use a single barometer for both pressure measurements.

Further remarks. The given error components correspond to the specification of the instrument, which are the same for every instrument of the same type. Since the value of the actual offset error is not known (can be anywhere within the given specification), corrections can not be applied to the result of the measurement. The actual scaling error can also be anywhere within the given interval (e.g. $\pm 0.1\%$), so this cannot be corrected either.

Chapter 3

Error calculation II.

3.1. The probability density function is the following:



The expected value is

$$E\{x\} = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} x \ 0.5 \ dx = \left[\frac{x^2}{4}\right]_{-1}^{1} = 0.$$

The standard deviation can be calculated using Steiner's theorem as

$$\sigma_x^2 = E\{x^2\} - E^2\{x\} = E\{x^2\},$$

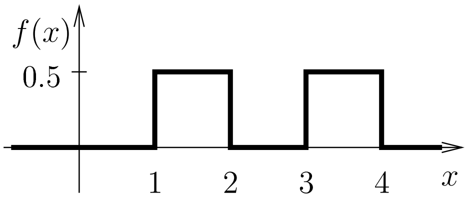
since the expected value is 0. The variance is

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_{-1}^{1} x^2 \ 0.5 \ dx = \left[\frac{x^3}{6}\right]_{-1}^{1} = \frac{1}{3},$$

so the standard deviation becomes

$$\sigma_x = \sqrt{\frac{1}{3}} = 0.5774.$$

3.2. The probability density function is the following:



The expected value can be read from the figure, since it is always the middle of the curve for symmetric distributions. It can also be calculated by the following integral:

$$E\{x\} = \int_{-\infty}^{\infty} xf(x)dx = \int_{1}^{2} x\ 0.5\ dx + \int_{3}^{4} x\ 0.5\ dx = \left[\frac{x^2}{4}\right]_1^2 + \left[\frac{x^2}{4}\right]_3^4 = 2.5.$$

The standard deviation can be determined using Steiner's theorem as

$$\sigma_x = \sqrt{E\{x^2\} - E^2\{x\}} = \sqrt{7.33 - 6.25} = 1.04,$$

since

$$E\{x^2\} = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{1}^{2} x^2\ 0.5\ dx + \int_{3}^{4} x^2\ 0.5\ dx = \left[\frac{x^3}{6}\right]_1^2 + \left[\frac{x^3}{6}\right]_3^4 = 7.33.$$

The interval in which the measured values can be found with 90% probability can be read from the figure: it is an interval in which the integral is 0.9. Any interval having the width $d = 2.8$ is appropriate within the interval $[1, 4]$. Thus, the bounds of the interval are $[1 + x,\ 3.8 + x]$, where $0 \leq x \leq 0.2$.

3.3. Since the distribution is normal, the 99.7% confidence level corresponds to $\pm 3\sigma$. This means that

$$\sigma = \frac{1}{6} = 0.1667.$$

3.4. The expected value and standard deviation of one blueberry is

$$\mu_1 = \frac{4.5 + 5.5}{2} \text{g} = 5 \text{ g}, \quad \sigma_1 = \frac{5.5 - 4.5}{\sqrt{12}} \text{g} = \frac{1}{\sqrt{12}} \text{ g}.$$

The standard deviation was determined by assuming uniform distribution. The expected value, variance and standard deviation of the weight of $N = 120$ blueberries are

$$\mu_{120} = N\mu_1 = 600 \text{ g}$$
$$\sigma_{120}^2 = N\sigma_1^2 = 120\frac{1}{12} \text{ g}^2 = 10 \text{ g}^2$$
$$\sigma_{120} = \sqrt{N}\sigma_1 = 3.162 \text{ g}.$$

Since we need to give a confidence interval with $p = 1 - b = 0.98 = 98\%$ probability, then, assuming a symmetric interval, $b/2 = 1\%$ is the probability that the random variable is higher than the upper bound $\mu_{120} + \Delta m$, and $b/2 = 1\%$ is the probability that it is smaller than the lower bound $\mu_{120} - \Delta m$. Since we were summing 120 independent random variables with uniform distribution, we can assume that the sum has a normal distribution. Therefore we need the $z_{b/2} = z_{0.01}$ value of a standard normal distribution which corresponds to 1% probability (the area between $z_{b/2}$ and $+\infty$ is $b/2 = 0.01$). However, the normal distribution table in the Appendix on page 101 lists probability values between 0 and $z_{b/2}$, so we actually look for a probability value of $1/2 - b/2 = 0.5 - 0.01 = 0.49$ in the table. This value is

$$z_{0.01} = 2.33.$$

Therefore the width of the symmetric confidence interval is

$$\Delta m = \sigma_{120} z_{0.01} = 7.3675 \text{ g}.$$

The confidence interval becomes:

$$P\left[\mu_{120} - \Delta m < m < \mu_{120} + \Delta m\right] = 98\%,$$

$$P\left[592.63 \text{ g} < m < 607.37 \text{ g}\right] = 98\%,$$

meaning that 98% of the cans have a weight between 592.63 g and 607.37 g, while only 1% are below, and 1% are above.

3.5. The expected value and variance of the random variable x_i which is uniformly distributed in $[0, a]$, $a = 1$:

$$\mu_1 = \frac{a}{2} = 0.5, \; \sigma_1^2 = \frac{a^2}{12} = 0.8333,$$

where μ_1 is the expected value and σ_1 is the standard deviation. Summing $N = 48$ independent samples:

$$\mu_N = N\mu_1 = 24, \qquad \sigma_N^2 = N\frac{a^2}{12} = \frac{48}{12} = 4.$$

Since we are summing a large number of independent uniformly distributed random variables, we can assume that the resulting distribution is normal. However, it is not a standard normal distribution, since its mean is not 0, and its standard deviation is not 1. To obtain standard normal distribution, the above variable has to be standardized. First, we subtract the expected value μ_N and by this we shift the probability density function (PDF) to the origin. Then we divide by σ_N which scales the PDF to have a unity standard deviation. Thus, the following operations are required:

First we sum the samples

$$x_N = \sum_1^N x_i$$

and then apply the operation

$$z = \frac{x_N - \mu_N}{\sigma_N} = \frac{x_N - 24}{2}$$

to get a standard normally distributed variable.

Remark I. The probability density function is a mathematical abstraction. In reality we have samples and the distribution of these samples converges to the theoretical PDF. If we want to change the distribution of a given set of samples, then we can only apply the transformation on the samples, and not on the (only theoretically existing) PDF. For example, if we want to shift the PDF of a variable to the left by 5, we have to subtract 5 from the random variable.

Remark II. Since the pseudo-random generator functions (e.g., rand function in C) in computers do indeed generate a uniformly distributed random number, this is a practical way to generate a normally distributed random variable.

3.6. The measured data is a constant value disturbed by a normally distributed noise with zero mean. This means that the constant value is actually the expected value of the measured data, and the best estimate for this constant is the average. Therefore, we will construct a confidence interval for the expected value.

The estimate of the expected value and the empirical standard deviation can be calculated by the following formulas:

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i = 14.5738, \qquad s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu})^2} = 4.7527.$$

where $\hat{\mu}$ is the estimate of the expected value, s is the empirical standard deviation, and $N = 6$ is the number of samples. The expected value and standard deviation of the average $\hat{\mu}$ are

$$E\{\hat{\mu}\} = \mu, \quad \sigma_{\hat{\mu}} = \frac{s}{\sqrt{N}},$$

since averaging decreases the standard deviation by $1/\sqrt{N}$. The standard deviation was estimated using the samples, therefore we have to use the Student-t distribution to determine the confidence level.

Since we want to determine a symmetric confidence interval with $1-b = 90\%$ probability, we need the value $t_{N-1,b/2}$ of the $N-1 = 5$ degrees of freedom Student-t distribution which the random variable can only exceed with $b/2$ probability (the area between $t_{N-1,b/2}$ and $+\infty$ is $b/2 = 0.05$). From the table on page 102 we obtain:

$$t_{(5,0.05)} = 2.015.$$

So the width of the symmetric confidence interval is

$$\Delta\hat{\mu} = \sigma_{\hat{\mu}} t_{(5,0.05)} = \frac{s}{\sqrt{N}} t_{(5,0.05)} = 3.9096.$$

and the confidence interval is

$$P\left[\hat{\mu} - \frac{s}{\sqrt{N}} t_{(5,0.05)} < \mu < \hat{\mu} + \frac{s}{\sqrt{N}} t_{(5,0.05)}\right] = 90\%.$$

After substitution:

$$P\left[10.6642 < \mu < 18.4835\right] = 90\%,$$

meaning that the constant value we are measuring is between 10.6642 and 18.4835 with 90% probability.

Remark. If the question were for an interval within the noisy data are with 90% probability, then we would have to construct the confidence interval for the data x_i and not for the expected value $\hat{\mu}$. In that case we don't divide s with $\sqrt{N}$ and thus $\Delta x = st_{(5,0.05)}$. So the confidence interval is:

$$P\left[\hat{\mu} - s\ t_{(5,0.05)} < \mu < \hat{\mu} + s\ t_{(5,0.05)}\right] = 90\%.$$

After substitution we get

$$P\,[4.9971 < \mu < 24.1505] = 90\%,$$

meaning that 90% of the random measured data are between 4.9971 and 24.1505. (We note that for $N = 6$ this is only true approximately, but for larger N the above procedure can be used to give a confidence interval for the data x_i with good accuracy.)

3.7. The 95.5% confidence level corresponds to a $\pm 2\sigma$ interval, therefore

$$\sigma_1 = \frac{\Delta x_1}{2} = 0.5 \text{ cm}, \quad \sigma_2 = \frac{\Delta x_2}{2} = 0.5 \text{ cm}, \quad \sigma_3 = \frac{\Delta x_3}{2} = 0.25 \text{ cm}.$$

The expected value and standard deviation of the total length of the three tables are

$$\begin{aligned} \mu_e &= \mu_1 + \mu_2 + \mu_3 = 300 \text{ cm}, \\ \sigma_e^2 &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \to \sigma_e = 0.75 \text{ cm}. \end{aligned}$$

Since the length of each table is a normally distributed random variable, the total length is also such a quantity. We have to determine the value that the total length of the three tables does not exceed with 99.7% probability. This condition limits only the maximal length of the three tables, and not the minimum, thus a one-side probability interval should be computed:

$$\begin{aligned} P\,[l < \mu_e + \sigma_e z_b] &= 1 - b, \\ P\left[l < \mu_e + \sigma_e z_{(0.003)}\right] &= 99.7\%. \end{aligned}$$

Note that we have used z_b instead of $z_{b/2}$ since we have a one-sided (non-symmetric) confidence interval.

The table of the standard normal distribution on page 101 gives the values for the positive side of the density function, thus, we are looking for $0.5 - 0.003 = 0.497$, which is:

$$z_{(0.003)} \approx 2.74.$$

As a result, the total length of the three tables is less than

$$\mu_e + \sigma_e z_{(0.003)} = 302 \text{ cm}$$

with 99.7% probability.

3.8. The errors have to be converted to standard uncertainties (standard deviations) by dividing with the extension factor z corresponding to the confidence level. Since the confidence level $1-b$ is 90%, and the interval is symmetric, we use $b/2 = 0.05$. From the table, $z_{b/2} = 1.64$, and we obtain

$$u(x) = \sigma_x = \frac{\Delta x}{z_{0.05}} = 6.1 \text{ m}, \quad u(t) = \sigma_t = \frac{\Delta t}{z_{0.05}} = 1.22 \text{ s}.$$

The velocity and the sensitivities are

$$v = \frac{x}{t}, \quad c_1 = \frac{\partial v}{\partial x} = \frac{1}{t}, \quad c_2 = \frac{\partial v}{\partial t} = -\frac{x}{t^2}.$$

The total standard uncertainty becomes

$$u(v) = \sigma_v = \sqrt{(c_1 u(x))^2 + (c_2 u(t))^2} = 0.0031 \text{ m/s}.$$

Thus, $\Delta v = ku(v) = 0.0062$ m/s. The velocity given in a standardized form is

$$v = 1.0000(62) \text{ m/s}.$$

3.9. The value of the standard uncertainties equal to the standard deviations:

$$u(U) = \sigma_U, \quad u(I) = \sigma_I.$$

The formulas of the resistance and the sensitivities are

$$R = \frac{U}{I}, \quad c_U = \frac{\partial R}{\partial U} = \frac{1}{I}, \quad c_I = \frac{\partial R}{\partial I} = -\frac{U}{I^2}.$$

So the value of the resistance and its standard uncertainty are

$$R = 1 \text{ k}\Omega, \quad u(R) = \sigma_R = \sqrt{(c_U u(U))^2 + (c_I u(I))^2} = 0.0141 \text{ k}\Omega.$$

3.10. The solution is similar to the problem of 3.5. The expected value of the given discrete distribution is obtained by summing the outcomes weighted by the probabilities:

$$\mu_1 = aP\{a\} \ + \ -aP\{-a\} = 2 \cdot 0.5 - 2 \cdot 0.5 = 0,$$

and the variance is

$$\sigma_1^2 = E\{(x-\mu_1)^2\} = (a-\mu_1)^2 P\{a\} + (-a-\mu_1)^2 P\{-a\} = a^2 = 4,$$

and thus the standard deviation is $\sigma_1 = 2$. We sum N independent samples to get x_N, for which we have

$$\mu_N = 0, \quad \sigma_N^2 = N\sigma_1^2 = 1024.$$

Thus, the sum x_N has zero mean, but the standard deviation is not unity, but $\sigma_N = 32$. Therefore, the sum has to be divided by the standard deviation

$$z = \frac{x_N}{\sigma_N} = \frac{x_N}{32}$$

to get standard normal distribution.

3.11. Student-t distribution has to be used since no a priori knowledge was available about the standard deviation, but it had to be estimated from the measured data.

a) The confidence interval has to determined for the average of the measurements, since we want an estimate for the nominal value of the coin. Therefore, the estimated standard deviation s_1 has to be divided by the square root of the number of samples. The confidence interval becomes

$$P\left[m_1 - \frac{s_1}{\sqrt{N}} t_{(N-1,b/2)} < m < m_1 + \frac{s_1}{\sqrt{N}} t_{(N-1,b/2)}\right] = 1 - b,$$

That is

$$P\,[2.9872 \text{ g} < m < 3.0128 \text{ g}] = 99\%.$$

b) For the second case, let's first write the confidence interval for the pack of $K = 40$ coins:

$$P\left[m_K - \frac{s_K}{\sqrt{N}} t_{(N-1,b/2)} < K\,m < m_K + \frac{s_K}{\sqrt{N}} t_{(N-1,b/2)}\right] = 1 - b.$$

This can be transformed into the confidence interval of one coin by dividing all the sides of the inequality by K:

$$P\left[\frac{m_K}{K} - \frac{s_K}{K\sqrt{N}} t_{(N-1,b/2)} < m < \frac{m_K}{K} + \frac{s_K}{K\sqrt{N}} t_{(N-1,b/2)}\right] = 1 - b,$$

After substitution, we obtain

$$P\,[2.99968 \text{ g} < m < 3.00032 \text{ g}] = 99\%.$$

Remark. We see that averaging decreases the standard deviation by a factor $1/\sqrt{N}$, while measuring K coins in the same time by a factor of $1/K$. This difference can be explained because in the averaging case we take N separate measurements with independent errors, and thus the total error (standard deviation) becomes $\sqrt{N}$ times larger when summing N measurements, which is then divided by N to give the average, and $\sqrt{N}/N = 1/\sqrt{N}$. On the other hand, when we measure K coins together, only one measurement is made with a single error, but this is still divided by K to get the mass of one coin.

A similar practical problem would be how to measure the weight of an A4 paper: here again it is better to measure a package of 500 sheets together and divide by 500, instead of measuring 500 sheets separately and averaging, not to mention that it takes much less time.

3.12. The error of each day can be calculated as the difference between the times displayed by the clocks at noon on consecutive days:

$$9 \quad 9 \quad 14 \quad 9 \quad 10 \quad 12 \quad [\text{sec}]$$

Since we are interested in the constant component of the total error (the systematic error), we need to estimate the expected value of the daily error. The estimates of the expected value and the standard deviation are

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i = 10.5 \text{ s}, \quad s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu})^2} = 2.0736 \text{ s}.$$

We have estimated the standard deviation, therefore we use a Student distribution in the confidence interval:

$$P\left[\hat{\mu} - \frac{s}{\sqrt{N}}t_{(N-1,b/2)} < \mu < \hat{\mu} + \frac{s}{\sqrt{N}}t_{(N-1,b/2)}\right] = 1 - b.$$

$$P\left[\hat{\mu} - \frac{s}{\sqrt{6}}t_{(5,0.025)} < \mu < \hat{\mu} + \frac{s}{\sqrt{6}}t_{(5,0.025)}\right] = 95\%.$$

where $t_{(5,0.025)} = 2.571$. That is

$$P\left[8.3235 \text{ s} < \mu < 12.6765 \text{ s}\right] = 95\%.$$

3.13.

a) Since we are measuring students, we have to use the Student distribution. ⌣̈. Seriously, the real reason why we use the Student distribution because we have estimated the standard deviation. The confidence interval is

$$P\left[\hat{\mu} - \frac{s}{\sqrt{N_1}}t_{(N_1-1,b/2)} < \mu < \hat{\mu} + \frac{s}{\sqrt{N_1}}t_{(N_1-1,b/2)}\right] = 1 - b,$$

$$P\left[\hat{\mu} - \frac{s}{\sqrt{9}}t_{(9,0.05)} < \mu < \hat{\mu} + \frac{s}{\sqrt{9}}t_{(9,0.05)}\right] = 90\%.$$

After substitution, we get

$$P\left[174.99 \text{ cm} < \mu < 181.01 \text{ cm}\right] = 90\%.$$

b) Here we utilize the fact that for large N Student distribution converges to normal distribution:

$$t_{(325,0.05)} \cong z_{(0.05)} = 1.64.$$

Thus, the confidence interval becomes

$$P\left[\hat{\mu} - \frac{s}{\sqrt{N_2}}z_{(0.05)} < \mu < \hat{\mu} + \frac{s}{\sqrt{N_2}}z_{(0.05)}\right] = 90\%,$$

which is

$$P\left[177.53 \text{ cm} < \mu < 178.47 \text{ cm}\right] = 90\%.$$

3.14. The estimate of the current is

$$\hat{I} = \frac{\hat{U}}{\hat{R}}.$$

The sensitivities are

$$\begin{aligned} c_U &= \frac{\partial I}{\partial U} = \frac{1}{R}, \\ c_R &= \frac{\partial I}{\partial R} = -\frac{U}{R^2}. \end{aligned}$$

a) To determine the type-A standard uncertainty of the voltage U first its standard deviation is estimated as

$$\hat{U} = \frac{1}{5}\sum_{i=1}^{5} U_i = 138.736 \text{ mV}, \quad s = \sqrt{\frac{1}{4}\sum_{i=1}^{5} (U_i - \hat{U})^2} = 33.616 \ \mu\text{V}.$$

Since we obtain the estimate of the voltage by averaging $N = 5$ measurements, the standard uncertainty is

$$u(U)_A = \frac{s}{\sqrt{5}} = 15.033 \ \mu\text{V}.$$

b) To determine the type-B standard uncertainty, first the specification of the manufacturer has to be studied. The quantization error is a part of the type-B uncertainty, so we don't have to consider it separately. Therefore we have

$$\Delta U_m = \hat{U} h_{o.v.} + h_{o.r.} U_{\max}.$$

We assume that the error is uniformly distributed. Therefore, the type-B standard uncertainty can be calculated using the expression of the standard deviation of a uniformly distributed random variable:

$$u(U)_B = \frac{\Delta U_m}{\sqrt{3}} = 21.793 \ \mu\text{V}.$$

c) The best estimate of the voltage is the average $\hat{U} = 138.736$ mV. The total uncertainty of U is the quadratic sum of the type-A and type-B uncertainties:

$$u(U) = \sqrt{u^2(U)_A + u^2(U)_B} = 26.475 \ \mu\text{V}.$$

To determine the actual value of the resistance a temperature corrections is needed. Its expected value and best estimate after the correction using the temperature-coefficient is:

$$\hat{R} = R_0[1 + \alpha(T - T_0)] = 100.131 \ \Omega,$$

where R_0 is the resistance at $T_0 = 20\ ^\circ\mathrm{C}$, R and T are the actual resistance and temperature, respectively. (Note that this correction will not be used when computing the uncertainty since it is the "error of the error".)

Only the type-B uncertainty can be calculated for the normal resistance. By assuming uniformly distributed measurement error we get

$$u(R) = u(R)_B = \frac{\Delta R}{\sqrt{3}} = 0.0266\ \Omega$$

where $\Delta R = 0.046\ \Omega$ is the random error of the resistance.

d) The estimate of the current and its standard uncertainty is

$$\hat{I} = \frac{\hat{U}}{\hat{R}} = 1.38554\ \mathrm{mA}, \quad u(I) = \sqrt{c_U^2 u^2(U) + c_R^2 u^2(R)} = 4.5323 \cdot 10^{-4}\ \mathrm{mA}.$$

The current given with its extended uncertainty with $k = 2$ is

$$I = 1.38554(91)\ \mathrm{mA}.$$

e) The extended uncertainty with $k = 2$ above represents a 95% confidence interval, under the assumption of normal distribution. The uncertainty has three sources in this problem: the two type-B uncertainties were determined assuming uniform distribution, the distribution of the type-A uncertainty of the voltage measurement is unknown, but it can be assumed as Gaussian. If more random variables are added, their probability density functions have to be convolved. The convolution of the above distributions is only approximately normal. In practice, the sum of 10-12 uniformly distributed random variables with a similar standard deviation can be assumed to be Gaussian.

f) The error of the voltmeter is a constant which does not change during the short time of the measurement. This error takes value inside the interval specified by the manufacturer. This interval was determined based on the parts of the instrument and observations during the testing. While the error is not random itself (it is constant), it is still treated as a random variable since we don't know where the actual error lies within the interval specified by the manufacturer. The randomness of the error can only be proven by taking measurements with more instruments of the same kind.

Chapter 4

Measurement of voltage and current

4.1. The definition of the signal-to-noise ratio is

$$\mathrm{SNR} = 10 \lg \frac{P_{\mathrm{signal}}}{P_{\mathrm{noise}}}.$$

So the ratio of the signal power and noise power is:

$$a = \frac{U_x^2}{U_n^2} = 10^{\mathrm{SNR}/10} \cong 29.51.$$

since $P_{\mathrm{signal}} = U_x^2$ and $P_{\mathrm{noise}} = U_n^2$. The RMS value of the measured signal U_m is the quadratic sum of the RMS values of the signal and noise components:

$$U_m^2 = U_x^2 + U_n^2 = U_x^2 + \frac{U_x^2}{a},$$

so the expression for the noiseless signal U_x becomes

$$U_x = \sqrt{\frac{U_m^2}{1 + 1/a}} = 6 \text{ V}.$$

4.2. The expected value is the mean value of the signal, thus, it is the same as the DC component. The RMS value can be calculated from the RMS value of the different frequency components with quadratic summation.

The DC component has zero frequency and its RMS value equals the DC value. The RMS value of a sine wave is its peak value scaled by $1/\sqrt{2}$. The fundamental frequency of the signal is the largest common divider of all frequency components (see also problems 1.5 and 1.6).

a)

$$x(t) = A^2 \sin^2(2\pi f_0 t) = A^2/2(1 - \cos(4\pi f_0 t)).$$

So we obtain

$$x_0 = A^2/2, \quad x_{\text{RMS}} = \sqrt{\left(\frac{A^2}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\frac{A^2}{2}\right)^2} = \sqrt{3/8}\,A^2, \quad f_x = 2f_0.$$

b)

$$x(t) = 1 \cdot \sin(3\pi f_0 t) - 0.9 \cdot \sin(3\pi f_0 t) = 0.1\sin(3\pi f_0 t)$$

$$x_0 = 0, \quad x_{\text{RMS}} = \frac{0.1}{\sqrt{2}} = 0.0707, \quad f_x = 1.5 f_0.$$

(The frequency is obtained by dividing the angular frequency $3\pi f_0$ by 2π.)

c)

$$x_0 = 0, \quad x_{\text{RMS}} = \sqrt{\left(\frac{12}{\sqrt{2}}\right)^2 + \left(\frac{12}{\sqrt{2}}\right)^2} = 12, \quad f_x = f_0.$$

d) Since for real signals $|x(t)|^2 = x^2(t)$, the absolute value sign does not influence the RMS value, therefore it is the same as for a normal sine wave ($1/\sqrt{2}$ times the peak value). Moreover, the mean value equals the absolute mean of a normal sine wave ($2/\pi$ times the peak value).

Since now the half-periods are equal because of the absolute sign, the time period of $x(t)$ is the half of the original sine wave. Thus, the frequency is the double.

$$x_0 = x_{\text{abs}} = \frac{2}{\pi}12 = 7.6394, \quad x_{\text{RMS}} = \frac{12}{\sqrt{2}} = 8.485, \quad f_x = 2f_0.$$

e)

$$x_0 = 0, \quad x_{\text{RMS}} = \sqrt{2} = 1.414, \quad f_x = f_0,$$

since for complex signals, the the absolute value is used to calculate the RMS value (we integrate $|z(t)|^2$), and $|z(t)| = \sqrt{2}$.

4.3.

$$U_1 = U_2 = \frac{U}{2} = 80 \text{ V}, \quad \Delta U_1 = \Delta U_2 = h_{\text{o.r.}} U_{\max}.$$

The worst case error is

$$\Delta U = \Delta U_1 + \Delta U_2 = 2h_{\text{o.r.}} U_{\max} = 2 \text{ V}.$$

The relative error is

$$\frac{\Delta U}{U} = \frac{2 \text{ V}}{160 \text{ V}} = 1.25\%.$$

4.4. Without any data about the errors of the instrument only the quantization error can be computed. Thus, we assume that the accuracy of the instrument corresponds to the accuracy of the display.

$$h \approx h_q = \frac{0.0001\ \text{V}}{0.0245\ \text{V}} = \frac{1}{245} \approx 0.4\%.$$

4.5.

a) The voltage of the components can be computed as

$$U_i\ [\text{V}] = 10^{U_i[\text{dB}]/20} = [1.000\ 0.2512\ 0.0631\ 0.0158\ 0.0040]\ \text{V}.$$

b) The RMS value is

$$U = \sqrt{\sum_{i=1}^{5} U_i^2[\text{V}]} = 1.033\ \text{V}.$$

c) The THD is

$$k = \frac{\sqrt{\sum_{i=2}^{5} U_i^2[\text{V}]}}{U} = \frac{\sqrt{U^2 - U_1^2[\text{V}]}}{U} = 25.12\%.$$

Note that dividing by the voltage of fundamental U_1 instead of the total RMS value U is also acceptable and it gives almost the same result since $U_1 \approx U$.

4.6.

a) The mean value of the signal is

$$U_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^{T_1} U_p dt = \frac{T_1}{T}U_p = 2\ \text{V}.$$

The RMS value is

$$U = \sqrt{\frac{1}{T}\int_0^T x(t)^2 dt} = \sqrt{\frac{1}{T}\int_0^{T_1} U_p^2 dt} = \sqrt{\frac{T_1}{T}}U_p = 3.162\ \text{V}.$$

The crest factor k_p and form factor k_f are

$$\begin{aligned} k_p &= \frac{U_p}{U} = \sqrt{\frac{T}{T_1}} = 1.581, \\ k_f &= \frac{U}{U_{\text{abs}}} = \frac{U}{U_0} = \sqrt{\frac{T_1}{T}\frac{T}{T_1}} = \sqrt{\frac{T}{T_1}} = 1.581, \end{aligned}$$

because the signal is never negative, the absolute mean value equals to the mean value $U_{\text{abs}} = U_0$.

b) The AC coupled measurement eliminates the DC component of the signal. This means that during the T_1 part of the period the value of the signal is $U_p - U_0$, while in the T_2 part its value is $-U_0$. The instrument may measure the positive or the negative peak of the signal, depending on its implementation. The displayed value is $1/\sqrt{2}$ times of the measured peak, since that is the correction factor for the sine wave. If the voltmeter measures the positive peak, we see

$$U_{\text{disp},1} = \frac{1}{\sqrt{2}}[U_p - U_0] = \frac{3\text{V}}{\sqrt{2}} = 2.121 \text{ V},$$

while if it measures the negative peak we read

$$U_{\text{disp},2} = \frac{1}{\sqrt{2}}U_0 = \frac{2\text{V}}{\sqrt{2}} = 1.414 \text{ V}.$$

4.7. For the square wave:

$$U_{\text{abs}} = U_p = U_{\text{RMS}} = 1 \text{ V}.$$

The displayed voltages for the absolute mean meter (U_1), the peak meter (U_2) and the true RMS meter (U_3) are the measured values multiplied by the correction factor of the instrument:

$$\begin{aligned} U_1 &= k_f U_{\text{abs}} = \frac{\pi}{2\sqrt{2}} U_{\text{abs}} = 1.111 \text{ V}, \\ U_2 &= \frac{U_p}{k_p} = \frac{U_p}{\sqrt{2}} = 0.7071 \text{ V}, \\ U_3 &= U_{\text{RMS}} = 1 \text{ V}. \end{aligned}$$

where k_f is the form factor and k_p is the crest factor of the sine wave.

4.8. The expression of the input resistance is:

$$R_{\text{in}} = \frac{U_1}{I_1}.$$

a) The error of the voltage and current measurement can be expressed as:

$$\begin{aligned} \frac{\Delta U_1}{U_1} &= h_{\text{o.v.}} + h_{\text{o.r.}} \frac{U_{\text{max}}}{U_1} + \frac{1}{N_{U_1}} = 0.066\%, \qquad (4.1) \\ \frac{\Delta I_1}{I_1} &= h_{\text{o.v.}} + h_{\text{o.r.}} \frac{I_{\text{max}}}{I_1} + \frac{1}{N_I} = 0.864\%, \end{aligned}$$

where $h_{\text{o.v.}}$ and $h_{\text{o.r.}}$ are the relative errors of value and of range, respectively, and $1/N_{U_1}$ and $1/N_I$ are the quantization errors for the voltage and current measurement ($N_{U_1} = 8765$ and $N_I = 172$). The total error using probabilistic summation becomes:

$$\frac{\Delta R_{\text{in}}}{R_{\text{in}}} = \sqrt{\left(\frac{\Delta U_1}{U_1}\right)^2 + \left(\frac{\Delta I_1}{I_1}\right)^2} = 0.87\%.$$

b) The expression of the input resistance is

$$R_{\text{in}} = \frac{U_2}{U_1 - U_2} R_s,$$

where U_2 is the voltage at the input after the connection of the potentiometer, and R_s is the value of the resistance of the potentiometer. If $U_2 = U_1/2$, then $R_{\text{in}} = R_s$.

Next, we calculate the error of R_{in}. The sensitivities are

$$\begin{aligned} c_{U_1} &= \frac{\partial R_{\text{in}}}{\partial U_1} = -\frac{R_s U_2}{(U_1 - U_2)^2}, \\ c_{U_2} &= \frac{\partial R_{\text{in}}}{\partial U_2} = \frac{R_s U_1}{(U_1 - U_2)^2}, \\ c_{R_s} &= \frac{\partial R_{\text{in}}}{\partial R} = \frac{U_2}{U_1 - U_2}, \end{aligned}$$

and thus the relative error components become

$$\begin{aligned} \left.\frac{\Delta R_{\text{in}}}{R_{\text{in}}}\right|_{U_1} &= c_{U_1} \frac{U_1}{R_{\text{in}}} \frac{\Delta U_1}{U_1} = -\frac{R_s U_2}{(U_1 - U_2)^2} U_1 \frac{U_1 - U_2}{R_s U_2} \frac{\Delta U_1}{U_1} = -\frac{U_1}{U_1 - U_2} \frac{\Delta U_1}{U_1}, \\ \left.\frac{\Delta R_{\text{in}}}{R_{\text{in}}}\right|_{U_2} &= c_{U_2} \frac{U_2}{R_{\text{in}}} \frac{\Delta U_2}{U_2} = \frac{R_s U_1}{(U_1 - U_2)^2} U_2 \frac{U_1 - U_2}{R_s U_2} \frac{\Delta U_2}{U_2} = \frac{U_1}{U_1 - U_2} \frac{\Delta U_2}{U_2}, \\ \left.\frac{\Delta R_{\text{in}}}{R_{\text{in}}}\right|_{R_s} &= c_{R_s} \frac{R_s}{R_{\text{in}}} \frac{\Delta R_s}{R_s} = \frac{U_2}{U_1 - U_2} R_s \frac{U_1 - U_2}{R_s U_2} \frac{\Delta R_s}{R_s} = \frac{\Delta R_s}{R_s}, \end{aligned}$$

so the error using probabilistic summation is

$$\frac{\Delta R_{\text{in}}}{R_{\text{in}}} = \sqrt{\left(\frac{\Delta R_s}{R_s}\right)^2 + 4\left(\frac{\Delta U_1}{U_1}\right)^2 + 4\left(\frac{\Delta U_2}{U_2}\right)^2} = 0.145\%,$$

where $\Delta U_2/U_2 = 0.0819\%$ can be calculated similarly to (4.1).

In the above calculations we have assumed that the error components are independent. Since the measurements are done with the same voltmeter, this might not be actually true. If the errors are not independent, and thus, summed with sign, they cancel because their opposite sign. Let's study the independence of the various error components separately!

1. *The "of value" error.* This is the gain error of the instrument which has the same value in the two cases so the summation with sign (as for systematic errors) should be used, and thus, these errors cancel each other. The condition for this is to use the instrument in the same range for both measurements. If the instrument was used in different ranges for the two measurements, this could not have been done!
2. *The "of range" error.* This can have more components: offset error, linearity error, and electrical noise. In the last two cases the errors are independent. Since there is no information about the proportion of the components, it is safer to assume that these errors do not cancel each other.

3. *The quantization error.* The exact value of this error depends on the method of quantization (rounding, truncation). The quantization errors are not completely independent since U_2 was set based on the value of U_1. However, they don't cancel out completely. Since it is better practice to overestimate the error than to underestimate it, we will take them into account in the final error calculation.

This means that we get a better estimate for the error if the "of value" errors are eliminated from the calculations, and only the "of range" and quantization errors are used. The modified errors are

$$\begin{aligned}\frac{\Delta U_1'}{U_1} &= h_{\text{o.r.}}\frac{U_{\max}}{U_1} + \frac{1}{N_{U_1}} = 0.016\%,\\ \frac{\Delta U_2'}{U_2} &= h_{\text{o.r.}}\frac{U_{\max}}{U_2} + \frac{1}{N_{U_2}} = 0.032\%.\end{aligned}$$

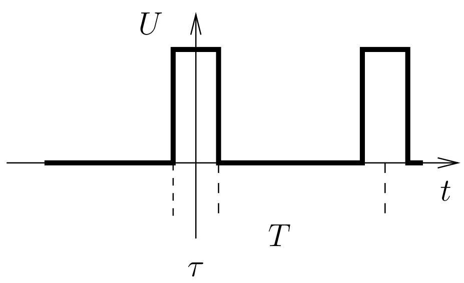
Thus, a more accurate estimate of the error is

$$\frac{\Delta R_{\text{in}}'}{R_{\text{in}}} = \sqrt{\left(\frac{\Delta R_s}{R_s}\right)^2 + 4\left(\frac{\Delta U_1'}{U_1}\right)^2 + 4\left(\frac{\Delta U_2'}{U_2}\right)^2} = 0.106\%.$$

4.9.

a) The Fourier series can be written in many ways, here we will show various cases. The square wave given in the problem has no starting phase, thus, we can shift it with respect to time. In general, if we are analyzing a signal in the frequency domain, the absolute phases of the components are not important, only their relative phase (phase differences). The phase is more important when analyzing more signals in parallel, e.g., in transfer function measurement, where the phase difference of the output and the input gives the phase shift of the transfer function.

1. *The real Fourier series with even symmetry.* By shifting the positive impulse of the signal into the origin $t = 0$ we get an even function, displayed below:



In the figure T is the period time, while τ is the length of the impulse.

The real Fourier series is defined as follows:

$$u(t) \cong U_0 + \sum_{k=1}^{\infty} U_k^A \cos k\omega t + \sum_{k=1}^{\infty} U_k^B \sin k\omega t, \quad \omega = \frac{2\pi}{T},$$

where U_0 is the mean value of the signal, computed as

$$U_0 = \frac{1}{T}\int_0^T U_p dt = \frac{\tau}{T} = 0.6 \text{ V},$$

where U_p is the peak voltage of the signal. Since the signal is represented by an even function, it has only cosine components, thus, $U_k^B \equiv 0$. The coefficients for the cosine can be computed by the integral

$$U_k^A = \frac{2}{T}\int_0^T \cos(k\omega t)u(t)dt = \frac{2}{T}\int_0^{\tau/2} \cos(k\omega t)U_p dt + \frac{2}{T}\int_{T-\tau/2}^{T} \cos(k\omega t)U_p dt =$$

$$= \frac{4U_p}{T}\int_0^{\tau/2} \cos k\omega t dt = \frac{2U_p}{k\pi}\sin\left(k\pi\frac{\tau}{T}\right),$$

where we have substituted $\omega = 2\pi/T$. Thus, we obtain

$$\begin{aligned} U_k^A \ &= \ [1.1226 \ \ 0.9082 \ \ 0.6055 \ \ 0.2806 \ \ 0.0 \\ &\quad -0.1871 \ \ -0.2595 \ \ -0.2270 \ \ -0.1247 \ \ 0.0] \text{ V}. \end{aligned}$$

Note that the 5th and 10th harmonics have zero amplitude, since the duty cycle of the square wave is $1/5$. This is because for a square wave with a duty cycle $1/k$ we are integrating a whole (or integer) period of sines and cosines with frequency $k\omega$, and the integral of an integer period is zero. This property is independent of the starting phase. The RMS values of the harmonics are computed as

$$U_{k,\text{RMS}} = U_k^A/\sqrt{2}, \quad k = 1, 2, \ldots,$$

since $U_k^B = 0$.

2. *The real Fourier series with the signal starting at the origin.* If we consider our signal so that the rising edge of the impulse is at $t = 0$, the Fourier coefficients become

$$\begin{aligned} U_k^{A'} &= \frac{2}{T}\int_0^T \cos(k\omega t)u(t)dt = \frac{2U_p}{T}\int_0^{\tau} \cos(k\omega t)dt = \frac{U_p}{k\pi}\left[-\sin\left(2k\pi\frac{t}{T}\right)\right]_0^{\tau}, \\ U_k^{B'} &= \frac{2}{T}\int_0^T \sin(k\omega t)u(t)dt = \frac{2U_p}{T}\int_0^{\tau} \sin(k\omega t)dt = \frac{U_p}{k\pi}\left[\cos\left(2k\pi\frac{t}{T}\right)\right]_0^{\tau}. \end{aligned}$$

By performing the calculations, we obtain

$$\begin{aligned} U_k^{A'} &= [0.9082 \ \ 0.2806 \ \ -0.1871 \ \ -0.2270 \ \ 0.0 \\ &\quad 0.1514 \ \ 0.0802 \ \ -0.0702 \ \ -0.1009 \ \ 0.0] \text{ V}, \\ U_k^{B'} &= [0.6598 \ \ 0.8637 \ \ 0.5758 \ \ 0.1650 \ \ 0.0 \\ &\quad 0.1100 \ \ 0.2468 \ \ 0.2159 \ \ 0.0733 \ \ 0.0] \text{ V}. \end{aligned}$$

where again the 5th and 10th components are missing. Now we have both sines and cosines, thus, the RMS values of the harmonics are

$$U_{k,\mathrm{RMS}} = \sqrt{\left(\frac{U_k^{A'}}{\sqrt{2}}\right)^2 + \left(\frac{U_k^{B'}}{\sqrt{2}}\right)^2}, \quad k = 1, 2, \ldots,$$

which are of course the same as the ones obtained from U_k^A in the even case.

3. *The complex Fourier series for the even case.* The complex Fourier series approximates the signal in the following way:

$$u(t) \cong \sum_{k=-\infty}^{+\infty} U_k^C e^{jk\omega t}.$$

Note that the index k runs from $-\infty$ to $+\infty$, and $k = 0$ corresponds to the DC component. In general, we obtain C_k as

$$C_k = \frac{1}{T} \int_0^T e^{-j\omega kt} u(t) dt.$$

However, since we have already computed the real Fourier series, it is easier to convert the real coefficients to complex ones. For the even case displayed in the figure, only the cosine components are nonzero, and they can be converted by the Euler formula as

$$U_k^A \cos(k\omega t) = \frac{U_k^A}{2} e^{j\omega kt} + \frac{U_k^A}{2} e^{-j\omega kt}.$$

Thus,

$$U_k^C = U_{-k}^C = \frac{U_k^A}{2}, \quad k = 1, 2, \ldots,$$

and

$$U_0^C = U_0.$$

The coefficients for positive and negative frequencies equal only because of the even symmetry. In general, U_{-k}^C is the complex conjugate of U_k^C for real signals. The RMS values of the harmonics are

$$U_{k,\mathrm{RMS}} = \frac{|U_k^C|}{\sqrt{2}}, \quad k = 1, 2, \ldots.$$

b) The RMS value can be computed according to the definition (root-mean-square integral). The true RMS meter displays the correct RMS value:

$$U_{\mathrm{disp,RMS}} = U_{\mathrm{RMS}} = \sqrt{\frac{1}{T} \int_0^T u(t)^2 dt} = U_p \sqrt{\frac{\tau}{T}} = 1.3416 \text{ V}.$$

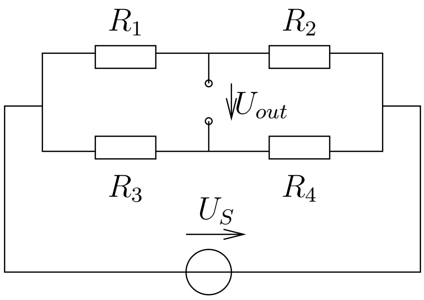
c) The lowpass filter lets through the components below $f_c = 5$ kHz. Since $f = 1/T = 2$ kHz, thus, only the DC component, and the first and the second harmonics have to be summed:

$$U_m = \sqrt{U_0^2 + \left(\frac{U_1^A}{\sqrt{2}}\right)^2 + \left(\frac{U_2^A}{\sqrt{2}}\right)^2} = 1.1843 \text{ V}.$$

Chapter 5

Measurement circuits

5.1. As an introduction to the problem, we provide a summary for the basic bridge circuits. The circuit can be seen in the following figure:



The output voltage of the bridge is the voltage difference between the two voltage dividers:

$$U_{\text{out}} = U_S \frac{R_2}{R_1 + R_2} - U_S \frac{R_4}{R_3 + R_4}.$$

All the various cases can be computed based on this basic equation.

Generally the resistances of the circuit have the same nominal value, where the fixed resistors are R, and the sensors are $R + \Delta R = (1 + h_R)R$ or $R - \Delta R = (1 - h_R)R$, where $h_R = \Delta R/R$ is the relative change, and R is the nominal value of the resistors. The behavior of the bridge depends on the number of sensors and on their positions. The type of excitation (voltage or current) also affects the behavior. The most important four cases are the following:

1. One resistance increases, the value of the other resistors do not change: the position of the sensor is arbitrary. Here we choose $R_2 = (1 + h_R)R$. The output voltage is

$$U_{\text{out}} = U_S \left(\frac{(1 + h_R)R}{R + (1 + h_R)R} - \frac{R}{R + R} \right) = U_S \frac{h_R}{4 + 2h_R} \cong \frac{U_T}{4} h_R;$$

2. Two resistors change in the same direction, these should be R_1 and R_4 or R_2 and R_3. Here we choose $R_2 = R_3 = (1 + h_R)R$. The output voltage is

$$U_{\text{out}} = U_S \left(\frac{(1 + h_R)R}{R + (1 + h_R)R} - \frac{R}{(1 + h_R)R + R} \right) = U_S \frac{h_R}{h_R + 2} \cong \frac{U_S}{2} h_R;$$

3. If the value of one resistor increases, another one decreases then they should be R_1 and R_2 or R_3 and R_4 to obtain linear behavior. Here we choose $R_2 = (1 + h_R)R$ and $R_1 = (1 - h_R)R$. The output voltage is

$$U_{\text{out}} = U_S \left(\frac{(1 + h_R)R}{(1 - h_R)R + (1 + h_R)R} - \frac{R}{R + R} \right) = \frac{U_S}{2} h_R;$$

4. If the value of two resistor increases and the value of the other two resistors decreases, then the value of the diagonally opposing resistors should change in the same direction, for example, we can choose $R_2 = R_3 = (1 + h_R)R$ and $R_1 = R_4 = (1 - h_R)R$. The output voltage is

$$U_{\text{out}} = U_S \left(\frac{(1 + h_R)R}{(1 - h_R)R + (1 + h_R)R} - \frac{(1 - h_R)R}{(1 + h_R)R + (1 - h_R)R} \right) = U_S h_R.$$

In the first two cases the output voltage is a nonlinear function of the input (the linear approximations are given with the $\cong$ sign). When a current source is used instead of the voltage supply, the output becomes linear for the second case as well. The output can be calculated as

$$U_{\text{out}} = \frac{I_S}{2}(R_2 - R_4) = \frac{I_S}{2}((1 + h_R)R - R) = \frac{I_S R}{2} h_R,$$

where I_S is the supply current.

Problem 5.1 corresponds to the second case with voltage supply. Since $\Delta R = 1\ \Omega$ and $h_R = 0.01$:

$$U_{\text{out}} = U_S \frac{h_R}{h_R + 2} = 24.9 \text{ mV}.$$

If the thermistors are connected with $2 \times 1\ \Omega$ wires to the bridge, the relative change of the resistances is $h'_R = 0.03$, so we get

$$U'_{\text{out}} = U_S \frac{h'_R}{h'_R + 2} = 73.9 \text{ mV}.$$

The systematic error is that part of the output voltage which corresponds to the voltage drop on the wires. Its value is

$$h_r = \frac{U'_{\text{out}} - U_{\text{out}}}{U_{\text{out}}} = 1.97 = 197\%.$$

We see that the wire resistance leads to an unacceptable error. Note, however, that because this is a systematic error, it can be measured and compensated. Another solution to the problem is to connect the ordinary resistors with wires having the same $2 \times 1\ \Omega$ resistance, so that the errors cancel.

5.2. The resistance and capacitance of the upper and lower parts of the divider are R_1, C_1, and R_2, C_2, respectively. The value $R_2 = 100$ kΩ is given, and in the compensated case the division ratio is

$$a = \frac{R_2}{R_1 + R_2} = 0.1.$$

Thus, the resistance of the upper part of the divider is

$$R_1 = 900 \text{ k}\Omega.$$

In the compensated case the time constants equal to each other:

$$R_1 C_1 = R_2 C_2.$$

Therefore, the capacitor connected in parallel with the upper resistor should be

$$C_2 = \frac{R_2 C_2}{R_1} = \frac{100}{9} \text{ pF} \approx 11.1 \text{ pF}.$$

5.3. The time function of the voltage appearing on the output of the multiplier is:

$$u_{\text{out}}(t) = kU_{\text{in,p}}^2 \sin^2 \omega t = k\frac{U_{\text{in,p}}^2}{2}(1 - \cos 2\omega t) = 5(1 - \cos 2\omega t) \text{ V}.$$

The mean value, absolute mean and RMS value can be determined based on problem 4.2:

$$\begin{aligned} U_0 &= 5 \text{ V}, \\ U_{\text{abs}} &= 5 \text{ V}, \\ U_{\text{RMS}} &= \sqrt{5^2 + 5^2/2} \text{ V} = 6.124 \text{ V}. \end{aligned}$$

5.4.

a) The circuit corresponds to the second case in problem 5.1. The value of the conventional resistors is $R = 100$ Ω, so that the output voltage at 25 °C is zero. We are using a current supply because that leads to a linear behavior.

b) The current supply equals to the sum of the currents of the two branches of the bridge:

$$I_S = 2\frac{U_{\text{t}}}{R} = 20 \text{ mA},$$

where $U_{\text{t}} = 1$ V is the voltage measured across the thermistor.

c)

$$\Delta R = R\alpha\Delta T = 0.3\ \Omega, \quad h_R = \frac{\Delta R}{R} = 0.003.$$

So the output voltage is

$$|U_{\mathrm{out}}| = \frac{I_S}{2}((1+h_R)R - R) = \frac{I_S R}{2} h_R = 3\ \mathrm{mV}.$$

d) The output voltage is linear function of the change of the value of the resistor, and

$$U_{\mathrm{out}} = \pm 5\ \mathrm{mV},\ \text{if } T = 0\ ^\circ\mathrm{C} \text{ or } T = 50\ ^\circ\mathrm{C}.$$

So we need an amplification of

$$A_U = \frac{10\ \mathrm{V}}{5\ \mathrm{mV}} = 2000.$$

5.5.

a) The circuit corresponds to the third case in problem 5.1.

b) The value of the ordinary resistors is $R = 400\ \Omega$, so that the output voltage is zero when there is no stress.

$$|U_{\mathrm{out}}| = U_S\left(\frac{(1+h_R)R}{(1-h_R)R + (1+h_R)R} - \frac{R}{R+R}\right) = \frac{U_S}{2} h_R = 10\ \mathrm{mV}.$$

c) The problem can be solved with the usual steps of error propagation computation (sensitivities, etc.). However, a simpler way is to consider what signs the resistance errors should have to give the largest output voltage error (worst case). This is similar to the problem of ordering the resistive sensors in a bridge so that the output voltage is maximal. This happens when the voltages at output points of the bridge change in the opposite direction. This is similar to the fourth case in problem 5.2, but the changes of the resistances have different values. Thus, the output voltage of the bridge is

$$|\Delta U| = U_S\left(\frac{(1+h_1)R}{(1-h_1)R + (1+h_1)R} - \frac{(1-h_2)R}{(1+h_2)R + (1-h_2)R}\right) =$$

$$= \frac{U_S}{2}(h_1 + h_2) = 35\ \mathrm{mV},$$

where $h_1 = 0.2\%$ and $h_2 = 0.5\%$ are the relative changes of the strain gauge resistors and the conventional resistors, respectively. Thus, the error of the measurement is

$$h = \frac{|\Delta U|}{|U_{\mathrm{out}}|} = 350\%.$$

5.6. By using the corresponding trigonometric formulas we get

$$u_{\mathrm{out}}(t) = kU_{\mathrm{in,p}}\sin(\omega t)U_{\mathrm{in,p}}\cos(\omega t) = k\frac{U_{\mathrm{in,p}}^2}{2}\cos(2\omega t) = 5\sin(2\omega t)\ \mathrm{V}.$$

And thus

$$\begin{aligned} U_0 &= 0\ \mathrm{V}, \\ U_{\mathrm{abs}} &= \frac{10}{\pi}\ \mathrm{V} = 3.183\ \mathrm{V}, \\ U_{\mathrm{RMS}} &= \frac{5}{\sqrt{2}}\ \mathrm{V} = 3.536\ \mathrm{V}. \end{aligned}$$

5.7. The relative phases of the signals are not given, but actually this does not matter for our case. Let's assume that they are the same. By using the corresponding trigonometric formula we obtain

$$u_{\mathrm{out}}(t) = kU_{\mathrm{p,1}}\sin(\omega_1 t)U_{\mathrm{p,2}}\sin(\omega_2 t) = k\frac{U_{\mathrm{p,1}}U_{\mathrm{p,2}}}{2}[\cos((\omega_1-\omega_2)t)-\sin((\omega_1+\omega_2)t)] =$$

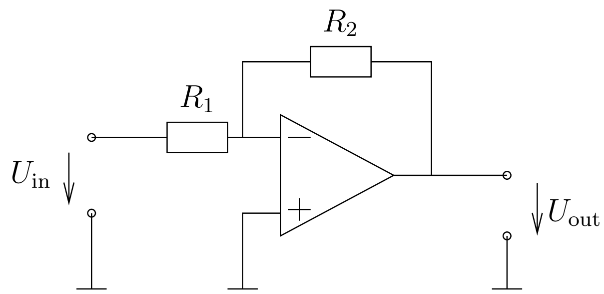
$$= 0.5\cos((\omega_1-\omega_2)t) - 0.5\cos((\omega_1+\omega_2)t)\ \mathrm{V}.$$

As a result,

$$\begin{aligned} U_0 &= 0\ \mathrm{V}, \\ U_{\mathrm{RMS}} &= \sqrt{\left(\frac{0.5}{\sqrt{2}}\right)^2 + \left(\frac{0.5}{\sqrt{2}}\right)^2}\ \mathrm{V} = 0.5\ \mathrm{V}. \end{aligned}$$

5.8.

a) The wiring diagram is the following:



The systematic error of the gain is computed as

$$A = -\frac{R_2}{R_1} = -5.1, \quad h_s = \frac{A - A_0}{A_0} = +2\%.$$

b) The new value of the feedback resistance is

$$R'_2 = R_2 \times R_3 = \frac{R_2 R_3}{R_2 + R_3} = 5.005\ \text{k}\Omega.$$

The new values of the gain and the systematic error are

$$A' = -\frac{R'_2}{R_1} = -5.0055, \quad h'_s = \frac{A' - A_0}{A_0} = +0.11\%.$$

c) The total error is the sum of the systematic error and the random error of the gain:

$$\frac{\Delta A}{A} = h'_s \pm |h_r|,$$

where the random error is

$$h_r = \left|\frac{\Delta R_1}{R_1}\right| + \left|\frac{\Delta R'_2}{R'_2}\right|,$$

since $A' = -R'^1_2 R_1^{-1}$ is the product of variables.

The error of R_1 is given, while the error of R'_2 has to be calculated from the errors of R_2 and R_3 which are connected in parallel. For the detailed solution with sensitivity calculation, see problem 2.5, from which we obtain

$$\left|\frac{\Delta R'_2}{R'_2}\right| = \left|\frac{R'_2}{R_2}\frac{\Delta R_2}{R_2}\right| + \left|\frac{R'_2}{R_3}\frac{\Delta R_3}{R_3}\right|$$

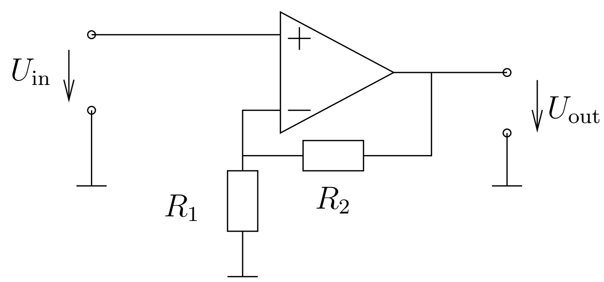
And thus the total worst case error is

$$\frac{\Delta A}{A} = h'_s \pm \left(\left|\frac{\Delta R_1}{R_1}\right| + \left|\frac{R_3}{R_2 + R_3}\frac{\Delta R_2}{R_2}\right| + \left|\frac{R_2}{R_2 + R_3}\frac{\Delta R_3}{R_3}\right|\right) = 0.11\% \pm 0.29\%,$$

which is an asymmetric error interval between -0.18% and 0.4%. If we want to describe the total error with a symmetric interval, thus, with a single number, then we choose the bound with the larger absolute value, that is, 0.4%.

5.9.

a) The wiring diagram is the following:



The systematic error of the gain is computed as

$$A = 1 + \frac{R_2}{R_1} = 10.1, \quad h_s = \frac{A - A_0}{A_0} = +1\%.$$

b) The value of the feedback resistance made from R_3 and R_4 is

$$R_2' = R_3 + R_4 = 9\ \text{k}\Omega.$$

The new values of the gain and the systematic error are

$$A' = \frac{R_1 + R_3 + R_4}{R_1} = 10, \quad h_r' = \frac{A' - A_0}{A_0} = 0.$$

c) The worst case error coming from the random errors can be determined using the usual error calculation. The sensitivities are

$$\begin{aligned} c_{R_1} &= \frac{\partial A'}{\partial R_1} = -\frac{R_3 + R_4}{R_1^2}, \\ c_{R_3} &= \frac{\partial A'}{\partial R_3} = \frac{1}{R_1}, \\ c_{R_4} &= \frac{\partial A'}{\partial R_4} = -\frac{1}{R_1}. \end{aligned}$$

The absolute error of A' is therefore

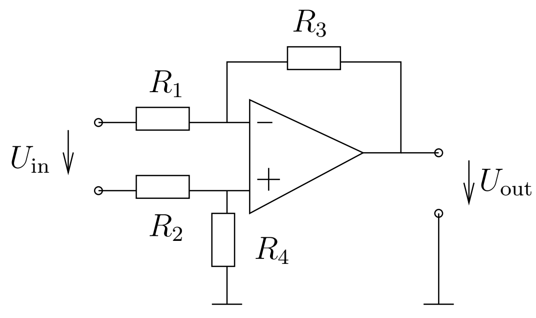
$$\Delta A' = \left| \Delta R_1 \frac{R_3 + R_4}{R_1^2} \right| + \left| \frac{\Delta R_3}{R_1} \right| + \left| \frac{\Delta R_4}{R_1} \right|.$$

The relative error is

$$\frac{\Delta A'}{A'} = \left| \Delta R_1 \frac{R_3 + R_4}{R_1 (R_1 + R_3 + R_4)} \right| + \left| \frac{\Delta R_3}{R_1 + R_3 + R_4} \right| + \left| \frac{\Delta R_4}{R_1 + R_3 + R_4} \right| = 0.27\%.$$

5.10.

a) The wiring diagram is the following:



The solution is fine even if the $R_1 - R_3$ and $R_2 - R_4$ pairs are exchanged.

b) By using the notations of the figure, the common gain is the following:

$$A_c = \frac{R_1 R_4 - R_2 R_3}{R_1 (R_2 + R_4)}.$$

The actual value of a single resistance is

$$R_i = R_{i,n}(1 \pm h),$$

where h is the maximum value of the relative deviation. Substituting this into the expression of A_c, and choosing the signs in $\pm$ so that A_c is maximal (since this is the worst case), we get

$$A_c \cong \frac{R_1 R_4 (1+h)^2 - R_2 R_3 (1-h)^2}{R_1(R_2+R_4)} = \frac{R_4}{R_2+R_4} 4h.$$

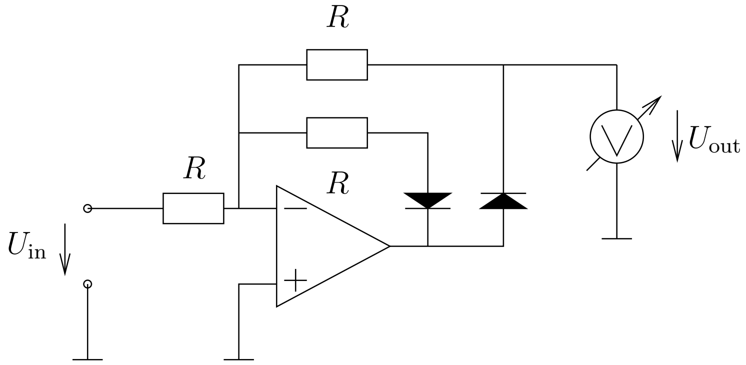
Note that the dependency of the denominator on the error was neglected and we utilized the fact that $R_1 R_4 \cong R_2 R_3$. The common-mode rejection ratio is

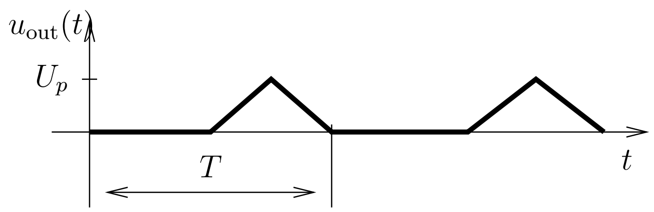
$$E = \frac{|A_s|}{|A_c|} = \frac{R_3}{R_1} \frac{R_2+R_4}{R_4 4h} = 12625 \cong 82 \text{ dB}.$$

The solution remains the same even if the exchanged resistor sets are used.

5.11.

a) The wiring diagram and the measured waveform can be seen below:





b) First the absolute mean of the signal has to be determined, which is the half of the absolute mean value in the case of a triangular wave:

$$U_{\text{abs}} = \frac{U_p}{4} = 0.25 \text{ V}.$$

The displayed value is

$$U_{\text{out}} = 0.25 \text{ V}.$$

c) The measured voltage is

$$U_m = -\frac{R_2}{R_1} A \frac{U_p}{4},$$

where A is a gain factor coming from the error of the voltmeter, determined by its accuracy class $h_{\text{o.r.}}$. The value of the latter is:

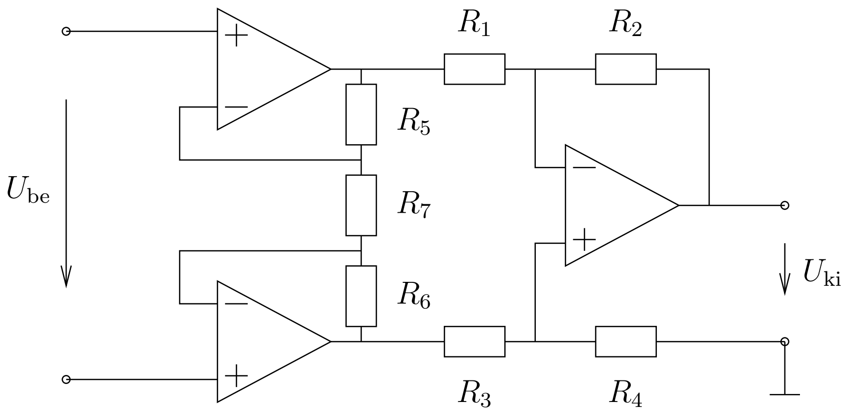
$$\frac{\Delta A}{A} = h_{\text{o.r.}} \frac{U_{\max}}{U_{\text{out}}} = 2\%.$$

We see that in the total measurement error the relative error of the two resistors R_1 and R_2 and the error caused by the voltmeter have a weight of 1 since they are either at power 1 or -1 in the expression of U_m. Therefore, the total measurement error assuming uniform distribution for all components and using $k = 2$ extension factor is

$$\frac{\Delta U_m}{U_m} = 2\sqrt{2\left(\frac{\Delta R}{R}\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{\Delta A}{A}\frac{1}{\sqrt{3}}\right)^2} = 2.83\%.$$

5.12.

a) The schematics can be seen below.



The values of the resistors are

$$R_1 = R_3 = 5\ \text{k}\Omega, \quad R_2 = R_4 = R_5 = R_6 = 25\ \text{k}\Omega, \quad R_7 = 5.55\ \text{k}\Omega.$$

b) The symmetric gain is

$$A_s = -\frac{R_2}{R_1}\left(1 + \frac{2R_5}{R_7}\right) = -50.045,$$

so the systematic error is

$$h_r = \frac{A_s - A_{s,0}}{A_{s,0}} = 0.09\%.$$

c) For computing the common-mode rejection ratio, first we compute the common-mode gain of the second stage (differential amplifier), which is

$$A_{c,2} = \frac{R_1 R_4 - R_2 R_3}{R_1(R_3 + R_4)}. \tag{5.1}$$

Here $R_1 = R_3 = R$, $R_2 = R_4 = 5R$, based on their nominal values. The actual values of the resistors are

$$R_1 = R(1 \pm h), \quad R_2 = 5R(1 \pm h), \quad R_3 = R(1 \pm h), \quad R_4 = 5R(1 \pm h),$$

where h is the maximal relative deviation from the nominal value. This can be substituted to (5.1), and for the worst case we obtain

$$A_{c,2} \cong \frac{5R^2(1+h)^2 - 5R^2(1-h)^2}{6R^2} = \frac{5R^2(4h)}{6R^2} = \frac{10}{3}h.$$

Since the common-mode gain of the first stage is $A_{c,1} = 1$, we have $A_c = A_{c,2}$. The common-mode rejection ratio is thus

$$E = \frac{|A_s|}{|A_c|} = 0.3 A_s \frac{1}{h} = 75000 = 97.5 \text{ dB}.$$

Chapter 6

Time and frequency measurement

6.1. The gate time is $t_{\mathrm{m}} = 10$ ms, and during this time the counter can measure maximum $N_{\max} = 10^5$ periods of the input signal. This corresponds to the frequency

$$f_{x,\max} = \frac{N_{\max}}{t_{\mathrm{m}}} = 10^7 \text{ Hz} = 10 \text{ MHz}.$$

6.2.

a) First a base angle is computed:

$$\varphi_0 = \arcsin \frac{b}{a} = 1.312 = 75.16°, \tag{6.1}$$

From this, the phase shit can be expressed as

$$\varphi = \begin{cases} \pm\varphi_0 & \text{, if the major axis is in the 1st and the 3rd quarter} \\ \pi \pm \varphi_0 & \text{, if the major axis is in the 2nd and the 4th quarter} \end{cases} .$$

b) The error components can be obtained from equation (6.1) by the usual rules of error calculation:

$$\begin{aligned} \Delta\varphi_0|_b &= \frac{r}{\sqrt{1-r^2}} \frac{\Delta b}{b}, \\ \Delta\varphi_0|_a &= -\frac{r}{\sqrt{1-r^2}} \frac{\Delta a}{a}, \end{aligned}$$

where $r = b/a$.
The reading errors are generally independent random variables with arbitrary sign. When using worst case error summation, the total error becomes

$$\Delta\varphi_0 = \frac{r}{\sqrt{1-r^2}} \left[\frac{\Delta a}{a} + \frac{\Delta b}{b} \right] = 0.1510 \approx 8.65°.$$

The above error calculation can be used until $r < 1$. If $r \approx 1$, the results are going to be wrong because $|\varphi_0| \leq \pi/2$, but this error interval might

enable an angle higher than $\pi/2$. Furthermore as $r \approx 1$ the errors are not independent. The error interval has to be limited in this case: $|\varphi_0| \leq \pi/2$.
Remark. The above expression results in an absolute error based on relative errors, and the multiplication factor is a value without unit (radian). Care had to be taken that the final result is not given in percents, but in radians.

6.3.

a) Since we do not have any information about the error of the clock, we assume that $\Delta f_0/f_0 = 0$. Therefore, the only error that occurs is due to the rounding when counting the periods of the input signal (quantization error), which is $\Delta N = \pm 1$ in absolute terms, and $\Delta N/N = \pm 1/N$ when relative errors are computed.

When one period is measured, the counter has the value of

$$N_1 = \frac{T_x}{T_0} = \frac{f_0}{f_x},$$

where $T_x = 1/f_x$ is the time period of the input signal and $T_0 = 1/f_0$ is the clock period. Thus, the measurement error is

$$h_1 = \frac{1}{N_1} = \frac{f_x}{f_0} = 1\%.$$

b) When measuring n periods, the counter has n times higher value:

$$N_n = \frac{nT_x}{T_0} = n\frac{f_0}{f_x} = nN_1,$$

thus, the error decreases n-times compared to h_1:

$$h_2 = \frac{1}{N_n} = \frac{1}{nN_1} = \frac{h_1}{n}.$$

In the problem h_2 is given, so we have to rearrange the above expression to obtain

$$n = \frac{h_1}{h_2} = 100.$$

c) Further reduction of the error needs statistical averaging, meaning that separate measurements are taken, their results are recorded and the numbers are arithmetically averaged. Arithmetic averaging of k samples decreases the standard deviation by a factor of $\sqrt{k}$. Thus, when averaging k measurements of the above type with error h_2, we get the following error:

$$h_3 = \frac{h_2}{\sqrt{k}}.$$

Since h_3 is given, we obtain k as

$$k = \frac{h_2^2}{h_3^2} = 100.$$

d) The previous expression can be used, but since now we use data from single-period measurements, the initial error is h_1, the required resulting error is $h_4 = 10^{-4}$, so

$$m = \frac{h_1^2}{h_4^2} = 10000.$$

The non-averaged measurements are disturbed by the quantization error, which has uniform distribution. Due to the averaging of many independent random variables, the averaged result will be normally distributed.

6.4. This corresponds to constant gate time measurement, where the measurement is done for full periods of the input signal. Therefore, in practice the actual gate time t_g can be longer than the required measurement time since we wait for the last period of the signal after t_m has passed. However, we neglect this small time difference when computing the measurement error, assuming $t_\mathrm{g} \cong t_\mathrm{m}$. Since in this problem the clock has no error, only quantization error can occur. In time period measurement, we are counting the clock cycles during the gate time t_g, thus, the counter has the value of

$$N = t_\mathrm{g} f_0 \cong t_\mathrm{m} f_0 = 10^6.$$

The relative error is therefore

$$\frac{\Delta f_x}{f_x} = \frac{\Delta T_x}{T_x} = \frac{1}{N} \cong \frac{1}{t_\mathrm{m} f_0} = 10^{-6}.$$

6.5.

a) The worst case error of the frequency measurement is

$$\frac{\Delta f_x}{f_x} = \frac{\Delta f_0}{f_0} + \frac{1}{N}$$

for both cases. The only difference is due to the quantization error $1/N$, and that method will be more accurate which results in a larger number N in the counter. When we are measuring frequency, we count the input signal f_x for t_m time, giving

$$N_f = t_\mathrm{m} f_x$$

as a counter value.

When measuring time period, we use constant gate time measurement to maximally utilize the available measurement time t_m. In this case we count the clock cycles f_0 for the gate time $t_\mathrm{g} \cong t_\mathrm{m}$:

$$N_t = t_\mathrm{g} f_0 \cong t_\mathrm{m} f_0.$$

Since for our case $f_0 > f_x$, we obtain $N_t > N_f$, thus, the quantization error will be smaller for the time period measurement. In general, whenever the clock frequency is higher than the frequency of the input signal, the time period measurement is more accurate (this is the case for many practical applications, since f_0 is in the order of few MHz). On the contrary, if $f_x > f_0$, the frequency metering principle should be used.

b) The error with time period measurement is

$$h_1 = \frac{\Delta f_0}{f_0} + \frac{1}{N_t} \cong \frac{\Delta f_0}{f_0} + \frac{1}{f_0 t_{\mathrm{m}}} = 5.01 \cdot 10^{-4}.$$

c)

$$h_2 = \frac{\Delta f_0}{f_0} + \frac{1}{N_t'} \cong \frac{\Delta f_0}{f_0} + \frac{1}{f_0 t_{\mathrm{m}}'} = 6 \cdot 10^{-6}.$$

The measurement error is very small in the second case. Therefore, the assumption of noiseless input is unrealistic. Furthermore, as many as 10000 periods of the signal fits the measurement time so the frequency of the signal may change during the measurement, thus it cannot be assumed constant. It might be that the frequency stability of the signal is less precise than the actual measurement.

6.6.

a) The error of the frequency measurement equals to the measurement error of the time period,

$$\frac{\Delta f_x}{f_x} = -\frac{\Delta T_x}{T_x},$$

since

$$f_x = \frac{1}{T_x}.$$

In general, the sign of the error does not matter, so we write

$$\frac{\Delta f_x}{f_x} = \frac{\Delta T_x}{T_x} \cong \frac{\Delta f_0}{f_0} + \frac{1}{N} = \frac{\Delta f_0}{f_0} + \frac{1}{t_{\mathrm{m}} f_0} = 3.02 \cdot 10^{-5}.$$

b) The phase shift can be calculated using the following expression:

$$\varphi = 2\pi \frac{\tau}{T_x} = 2\pi\tau f_x, \tag{6.2}$$

where τ is the time delay between the two signals. So the absolute error of the phase measurement is

$$\Delta\varphi = \varphi \left[\frac{\Delta(f_x)'}{f_x} + \frac{\Delta\tau'}{\tau} \right].$$

The explanation of the $\Delta(f_x)'$ and $\Delta\tau'$ notations is the following: The clock cycle of the instrument causes the same relative error in the measurement of τ and T_x (assuming stable frequency), so this error is canceled in equation (6.2). Nevertheless, there are other independent error sources in the measurement of τ and T_x which present in the above equation. These are denoted with commas since the values of these errors differ from the total measurement errors of τ and T_x.

During the measurement time the τ interval and T_x are both measured $n = [t_\mathrm{m} f_x] \cong t_\mathrm{m} f_x$ times. However, the τ intervals are separated, so (assuming that f_x and f_0 are not synchronized) n independent measurements are available, and only $\sqrt{n}$-fold decrease occurs in the error during the averaging instead of the n-fold decrease of the average time period measurement. So the error of the measurement of τ is:

$$\frac{\Delta\tau'}{\tau} = \frac{1}{\sqrt{n}}\frac{1}{\tau f_0} = \frac{1}{\sqrt{t_\mathrm{m} f_x}}\frac{1}{\tau f_0}.$$

and for the frequency, it is:

$$\frac{\Delta f_x'}{f_x} \cong \frac{1}{t_\mathrm{m} f_0}.$$

Thus, the total error of the phase measurement is

$$\Delta\varphi = \varphi\left[\frac{1}{t_m f_0} + \frac{1}{\sqrt{t_m f_x}}\frac{1}{\tau f_0}\right] = 1.379 \cdot 10^{-5}\ \mathrm{rad} = 7.903 \cdot 10^{-4}\ {}^\circ. \quad (6.3)$$

c) Stopping the time-delay measurement with the falling edge of the output signal means that we are measuring a larger time delay and larger phase difference, so at first glance we would assume that this decreases the error. This indeed decreases the relative error since we are dividing by a larger φ value, but the question was about the absolute error $\Delta\varphi$, which leads to a different result.

The absolute error of the phase measurement from equation (6.2) and (6.3) is

$$\Delta\varphi = 2\pi\tau f_x\left[\frac{1}{t_m f_0} + \frac{1}{\sqrt{t_m f_x}}\frac{1}{\tau f_0}\right] = \frac{2\pi\tau f_x}{t_m f_0} + \frac{1}{\sqrt{t_m f_x}}\frac{2\pi f_x}{f_0}.$$

The measurement error of the time delay is independent from τ (second term above), and the absolute error coming from the frequency measurement depends on τ (first term above). Thus, larger τ corresponds to a larger error, so this second method actually decreases the precision of the phase measurement.

Chapter 7

Impedance and power measurement

7.1. The bridge is balanced when the voltages are the same at the two inputs of the voltmeter:

$$\frac{Z_1}{Z_1+Z_2}U_g = \frac{Z_3}{Z_3+Z_4}U_g. \tag{7.1}$$

This leads to the condition of balance

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4},$$

or, equivalently,

$$\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4}. \tag{7.2}$$

Note that it is worthy to solve such problems by starting from Eq. (7.2) instead of Eq. (7.1). Substituting the imdepance values into Eq. (7.1) would lead to much more tedious calculations compared to the relatively simple derivations below.

a) In our case, $Z_1 = Z_x = 1/(G_x + 1/j\omega G_x)$, $Z_2 = R_2$, $Z_3 = R_3$, and $Z_4 = R_4 + 1/j\omega C_4$. By substituting these into the balance condition Eq. (7.2) we obtain

$$\frac{1}{R_3(G_x + 1/j\omega L_x)} = \frac{R_2}{R_4 + 1/j\omega C_4}.$$

Since the above equation is complex, both the real and imaginary parts must equal. After flipping the fractions the equations for the real and imaginary parts become

$$\begin{aligned} R_3 G_x &= \frac{R_4}{R_2} \\ \frac{R_3}{j\omega L_x} &= \frac{1}{j\omega C_4 R_2} \end{aligned}$$

Thus, the elements of the measured impedance are

$$G_x = \frac{R_4}{R_2 R_3} = 100\ \mu\text{S}, \quad (R_x = 10\ \text{k}\Omega), \quad L_x = C_4 R_2 R_3 = 100\ \text{mH}.$$

b) In the case of $\omega' = 2000$ 1/s the elements of the impedance are the following:

$$G'_x = \frac{R'_4}{R_2 R_3} = 25\ \mu\text{S}, \quad L'_x = C'_4 R_2 R_3 = 100\ \text{mH}.$$

Two-parameter impedance models can be parallel or series RL or RC elements, thus, 4 different equivalent circuits exist. The simplest way to see whether the model is acceptable is to check the sign of the reactive element: a negative capacitance means that an inductive model should be used instead, and a negative inductance shows that a capacitive model should be used. In addition, a good model should be valid at a wide range of frequencies meaning that the parameters of the model should be frequency independent. In our problem $L_x = L'_x > 0$, thus, we need an inductive model indeed. However, $G'_x = G_x/4$ means that the parameters of the model are different at the two measurement frequencies. Therefore, we may try to convert the parallel RL model to a series RL model and check how the new parameters depend on frequency. The series RL model Z_s can be related to the parallel one Y_x as

$$\begin{aligned} \frac{1}{Y_x} &= Z_s, \\ \frac{1}{G_x + 1/j\omega L_x} &= R_s + j\omega L_s. \end{aligned}$$

The left hand side contains j in the denominator, so we multiply with the complex conjugate:

$$\frac{G_x - 1/j\omega L_x}{(G_x - 1/j\omega L_x)(G_x + 1/j\omega L_x)} = \frac{G_x}{G_x^2 + 1/\omega^2 L_x^2} - \frac{1/j\omega L_x}{G_x^2 + 1/\omega^2 L_x^2}$$

Then, equating the first term with R_s and the second term with $j\omega L_s$ we obtain

$$\begin{aligned} R_s &= \frac{\omega^2 G_x L_x^2}{1 + \omega^2 G_x^2 L_x^2} = 0.9999\ \Omega, \\ L_s &= \frac{L_x}{1 + \omega^2 G_x^2 L_x^2} = 99.99\ \text{mH}. \end{aligned}$$

For the higher measurement frequency ω' we get

$$\begin{aligned} R'_s &= \frac{\omega'^2 G'_x L'^2_x}{1 + \omega'^2 G'^2_x L'^2_x} = 0.99975\ \Omega, \\ L'_s &= \frac{L'_x}{1 + \omega'^2 G'^2_x L'^2_x} = 99.975\ \text{mH}. \end{aligned}$$

Since $R_s \cong R'_s$ and $L_s \cong L'_s \cong L_x$, the series RL model is better than the parallel one.

7.2. The quality factor Q_f is the ratio of the reactive and active power on the impedance. For a series model we assume that the current I is known and thus we can write

$$Q_f = \frac{Q}{P} = \frac{I^2 \omega L_s}{I^2 R_s} = \frac{\omega L_s}{R_s},$$

where L_s and R_s are the elements of the series RL model. Note that in the general case, the following expression is also true:

$$Q_f = \frac{\mathrm{Im}\{Z\}}{\mathrm{Re}\{Z\}} = \frac{\mathrm{Im}\{Y\}}{\mathrm{Re}\{Y\}},$$

where $\mathrm{Re}\{Z\}$ and $\mathrm{Im}\{Z\}$ are the real and imaginary parts of Z. A good quality inductance or capacitor have small dissipation, thus, small P compared to Q, meaning that Q_f is large.

The loss factor $\mathrm{tg}\delta$ equals to the dissipation ratio D, and it is the reciprocal of Q_f:

$$D = \mathrm{tg}\delta = \frac{P}{Q} = \frac{1}{Q_f} = \frac{R_s}{\omega L_s}.$$

We compute the values of the equivalent circuits such that at the measurement frequency the real and imaginary parts of the two models should be equal. The impedance and admittance of the series RL circuit are

$$\begin{aligned} Z_{L,s} &= R_s + j\omega L_s, \\ Y_{L,s} &= \frac{1}{R_s + j\omega L_s} = \frac{R_s - j\omega L_s}{R_s^2 + \omega^2 L_s^2} = \frac{1}{R_s}\frac{1 - j\omega L_s/R_s}{1 + \omega^2 L_s^2/R_s^2} = \frac{1}{R_s}\frac{1 - jQ_f}{1 + Q_f^2}. \end{aligned} \tag{7.3}$$

First we compute the parameters of the parallel RL model. For parallel models, it is more straightforward to work with admittance instead of impedance. Thus, the admittance of the parallel RL model is

$$Y_{L,p} = \frac{1}{R_p} + \frac{1}{j\omega L_p},$$

and now the task is to make the real and imaginary parts equal to the real and imaginary parts of Eq. (7.3):

$$\begin{aligned} \frac{1}{R_p} &= \frac{R_s}{R_s^2 + \omega^2 L_s^2}, \\ \frac{1}{j\omega L_p} &= -\frac{j\omega L_s}{R_s^2 + \omega^2 L_s^2}. \end{aligned}$$

After rearrangement, we obtain the parameters of the parallel RL model as

$$\begin{aligned} R_p &= R_s\left(1 + \omega^2 \frac{L_s^2}{R_s^2}\right) = R_s(1 + Q_f^2), \\ L_p &= L_s \frac{1 + \omega^2 L_s^2/R_s^2}{\omega^2 L_s^2/R_s^2} = L_s \frac{1 + Q_f^2}{Q_f^2} = L_s(1 + D^2). \end{aligned}$$

In the case of low loss factor (high quality factor) $L_p \approx L_s$.

Next, we compute the parameters of the series RC model. For series models, working with impedances leads to simpler equations. Thus, we write

$$Z_{L,s} = R_s + j\omega L_s \quad = \quad Z_{C,s} = R_{C,s} + \frac{1}{j\omega C_s}.$$

We make the real and imaginary parts equal to obtain

$$\begin{aligned} R_{C,s} &= R_s, \\ C_s &= -\frac{1}{\omega^2 L_s}. \end{aligned}$$

Note that the capacitance is negative.

For the parallel RC circuit we utilize the fact that we have already computed a parallel RL model, thus, we make the real and imaginary parts of the admittance for these two models equal:

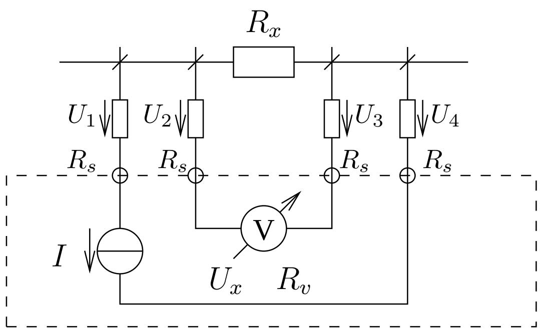
$$Y_{L,p} = \frac{1}{R_p} + \frac{1}{j\omega L_p} \quad = \quad Y_{C,p} = \frac{1}{R_{C,p}} + j\omega C_p.$$

Then we obtain

$$\begin{aligned} R_{C,p} &= R_p = R_s\left(1 + \omega^2 \frac{L_s^2}{R_s^2}\right) = R_s(1 + Q_f^2), \\ C_p &= -\frac{1}{\omega L_p^2} = -\frac{L_s}{R_s^2 + \omega^2 L_s^2}. \end{aligned}$$

The capacitance is negative again. In the case of low loss factor (high quality factor) $C_p \approx C_s$.

7.3. The wiring diagram is the following:



With the notations of the figure we have

$$\begin{aligned} U_1 = U_4 &= IR_s = 10 \text{ mV}, \\ U_2 = U_3 &= \frac{U_x}{R_v} R_s = \frac{IR_x}{R_v} R_s = 100 \text{ nV}. \end{aligned}$$

The voltage drop on the measured resistance is $U_x = 100$ mV. When using the two-wire measurement, the voltmeter measures $U_m = U_1 + U_x + U_2 = 120$ mV, leading to a 20% measurement error. For four-wire measurement, $U_m = U_2 + U_x + U_3 = 100.0001$ mV, showing that the error due to the voltage drop on the wires is practically eliminated.

7.4. The resistance of the measuring leads does not affect the result in the case of four-wire measurement. The effect of stray capacitances can be neglected since the frequency is quite low. Thus, the error depends only on the errors of the voltage and current measurement:

$$\frac{\Delta R}{R} = \frac{\Delta U}{U} + \frac{\Delta I}{I} = 1\%.$$

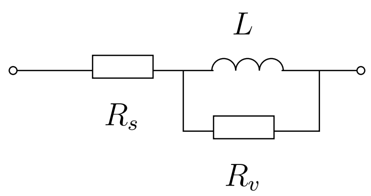
7.5. The measuring leads cause systematic error, since the three-wire technique does not cancel the error due to wire resistance. The signs of the systematic and random errors is the same in the worst case, thus the total error is

$$\frac{\Delta R}{R} = h_r + \frac{\Delta U}{U} + \frac{\Delta I}{I} = \frac{2R_s}{R} + \frac{\Delta U}{U} + \frac{\Delta I}{I} = 3\%.$$

7.6. In theory, the five-wire measurement cancels the errors both coming from the stray impedances and wire resistance. At 10 kHz frequency this is true also in the practice. So the measurement error comes only from the inaccuracy of voltage and current measurements:

$$\frac{\Delta R}{R} = \frac{\Delta U}{U} + \frac{\Delta I}{I} = 1\%.$$

7.7. The three parameter model of the magnetic-core coil can be seen in the following figure:



where R_s and R_v represent the copper and core losses, respectively, and L is the inductance. At DC, R_v is shunted by L, thus, the DC ohmmeter measurement gives the value of R_s directly:

$$R_s = 0.5\ \Omega.$$

The impedance of the three parameter model is

$$Z_3 = R_s + \frac{j\omega L R_v}{j\omega L + R_v} = R_s + \frac{j\omega L R_v (R_v - j\omega L)}{\omega^2 L^2 + R_v^2} =$$

$$= \left(R_s + \frac{\omega^2 L^2 R_v}{\omega^2 L^2 + R_v^2} \right) + j\omega \left(\frac{L R_v^2}{\omega^2 L^2 + R_v^2} \right),$$

where the expression in the first parenthesis equals to R_e and the one in the second parenthesis equals to L_e since the impedance of the series RL model is

$$Z_e = R_e + j\omega L_e.$$

After some maths, the parameters can be expressed as

$$R_v = R'_h \frac{r^2 + \omega^2}{r^2} = 999.5\ \Omega,$$
$$L = \frac{rR_v}{\omega^2} = 20.00\ \text{mH};$$

where

$$r = \frac{R'_e}{L_e};\ R'_e = R_e - R_s.$$

7.8. The impedance meter measures the voltage between the "high" node of R_x and the ground, and the current flowing from the "low" node of R_x to the ground. These are U_m and I_m, respectively. The resistor R_x is estimated by the ratio of them:

$$\hat{R}_x = \frac{U_m}{I_m}. \tag{7.4}$$

We first compute I_x, that is the current of R_x, assuming U_m excitation voltage to the measurement circuit:

$$I_x = \frac{U_m}{R_x + R_A \times R_g}.$$

However, the ammeter measures less current, as a small portion of the current flows to the ground through R_g. Therefore the measured current is

$$I_m = I_x \frac{R_g}{R_A + R_g} = \frac{U_m}{R_x + R_A \times R_g} \cdot \frac{R_g}{R_A + R_g}.$$

Substituting this into Eq. (7.4), after some algebra we get

$$\hat{R}_x = R_x \left(\frac{R_A + R_g}{R_g} + \frac{R_A}{R_x} \right).$$

a) Thus the relative error of the measurement is

$$\frac{\Delta R_x}{R_x} = \frac{\hat{R}_x - R_x}{R_x} = \frac{R_A + R_g}{R_g} + \frac{R_A}{R_x} - 1 = \frac{R_A}{R_g} + \frac{R_A}{R_x} = 1.1\%.$$

b) The above expression shows that the non-ideal ammeter is responsible for the systematic error of the measurement of R_x. Nevertheless, only the first item depends on R_g, that is the error caused by the R_g resistors is the following:

$$\left.\frac{\Delta R_x}{R_x}\right|_{R_g} = \frac{R_A}{R_g} = 0.1\%.$$

Note that this error does not depend on the value of R_x.

7.9.

a) The active power P and $\cos\varphi$ are

$$P = \frac{U_G^2 - U_Z^2 - U_R^2}{2R} = 163.6 \text{ mW}, \quad \cos\varphi = \frac{U_G^2 - U_Z^2 - U_R^2}{2U_Z U_R} = 0.4863.$$

b) The relative errors of the voltage measurements can be expressed using the accuracy class $h_{\text{o.r.}} = 0.5\%$:

$$h_G = \frac{U_{\max}}{U_G} h_{\text{o.r.}}, \quad h_Z = \frac{U_{\max}}{U_Z} h_{\text{o.r.}}, \quad h_R = \frac{U_{\max}}{U_R} h_{\text{o.r.}}.$$

The sensitivities for P are

$$\begin{aligned} c_G &= \frac{\partial P}{\partial U_G} = \frac{2U_G^2}{U_G^2 - U_Z^2 - U_R^2}, \\ c_Z &= \frac{\partial P}{\partial U_Z} = \frac{2U_Z^2}{U_G^2 - U_Z^2 - U_R^2}, \\ c_R &= \frac{\partial P}{\partial U_R} = \frac{2U_R^2}{U_G^2 - U_Z^2 - U_R^2}. \end{aligned}$$

The error of the power measurement with $k = 2$ extension factor, by the assumption of uniform distribution of voltage errors is

$$\frac{\Delta P}{P} = 2\sqrt{c_G^2\left(\frac{h_G}{\sqrt{3}}\right)^2 + c_Z^2\left(\frac{h_Z}{\sqrt{3}}\right)^2 + c_R^2\left(\frac{h_R}{\sqrt{3}}\right)^2} = 4.56\%.$$

The sensitivities for $\cos\varphi$ are

$$\begin{aligned} q_G &= \frac{\partial \cos\varphi}{\partial U_G} = \frac{U_G^2}{U_Z U_R}, \\ q_Z &= \frac{\partial \cos\varphi}{\partial U_Z} = \frac{U_R^2 - U_G^2 - U_Z^2}{2U_Z U_R}, \\ q_R &= \frac{\partial \cos\varphi}{\partial U_R} = \frac{U_Z^2 - U_G^2 - U_R^2}{2U_Z U_R}. \end{aligned}$$

The measurement error of $\cos\varphi$ with $k = 2$ extension factor, by the assumption of uniform distribution of voltage errors is

$$\Delta\cos\varphi = 2\sqrt{q_G^2\left(\frac{h_G}{\sqrt{3}}\right)^2 + q_Z^2\left(\frac{h_Z}{\sqrt{3}}\right)^2 + q_R^2\left(\frac{h_R}{\sqrt{3}}\right)^2} = 0.03053.$$

c) The sign of φ is not specified in the measurement, thus the type of the load cannot be determined.

7.10.

a) The absolute value and phase of the impedance are

$$|Z| = \frac{U_x}{U_N} R_N = 97.50\ \Omega, \quad \varphi = \arccos \frac{U_g^2 - U_x^2 - U_N^2}{2U_x U_N} = 1.5406 = 88.25°.$$

b) The error of $|Z|$ can be estimated based on the quantization errors:

$$\frac{\Delta|Z|}{|Z|} = \frac{\Delta R_N}{R_N} + \frac{\Delta U_x}{U_x} + \frac{\Delta U_N}{U_N} = 0.01\% + \frac{1}{7053} + \frac{1}{6877} = 3.87 \cdot 10^{-4} \approx 0.04\%.$$

c) Since the expression of $\cos\varphi$ contains differences, the method will be very sensitive to the errors of the voltage measurement whenever $\cos\varphi \approx 0$ (i.e., $\varphi \approx 90°$). Since this case occurs in the problem, the phase will be inaccurate, thus, the amplitude measurement will be more precise.

7.11.

a) The solution is similar to that of problem 7.1. The condition of balance is:

$$\begin{aligned} \frac{Z_x}{Z_3} &= \frac{Z_2}{Z_4}, \\ \frac{R_x + j\omega L_x}{R_3} &= R_2(G_4 + j\omega C_4). \end{aligned}$$

After making the real and imaginary parts equal, the elements of the impedance are obtained as

$$R_x = \frac{R_2 R_3}{R_4} = 1\ \Omega, \quad L_x = R_2 R_3 C_4 = 5\ \text{mH}.$$

b) The quality factor is

$$Q_f = \frac{Q}{P} = \frac{\omega L_x I^2}{R_x I^2} = \frac{2\pi f L_x}{R_x} = 5.$$

c) The easiest way to take the loss factor of C_4 into account is to model C_4 with a parallel RC circuit, where an R_p resistance is connected in parallel with C_4. The loss factor for this circuit is

$$D = \frac{P}{Q} = \frac{U^2/R_p}{U^2 \omega C_4} = \frac{1}{\omega R_p C_4}.$$

From this, the parallel resistor is

$$R_p = \frac{1}{D_4 2\pi f C_4} = 1\ \text{M}\Omega.$$

This R_p resistor is connected in parallel with C_4 and R_4. When the bridge is balanced, we read R_4, however, its real value is the parallel connection of the resistors, that is,

$$R_4' = R_4 \times R_p \cong 9901\ \Omega.$$

Thus, the real resistance of the series model is

$$R_x' = \frac{R_2 R_3}{R_4'} \cong 1.01\ \Omega,$$

so the error is

$$\frac{\Delta R_x}{R_x} = \frac{R_x' - R_x}{R_x} = 1\%.$$

7.12.

a) The condition of balance is

$$\frac{Z_N}{Z_x} = \frac{Z_4}{Z_3},$$
$$j\omega C_N(R_x + 1/j\omega C_x) = (j\omega C_4 + G_4)R_3.$$

The elements of the measured impedance are

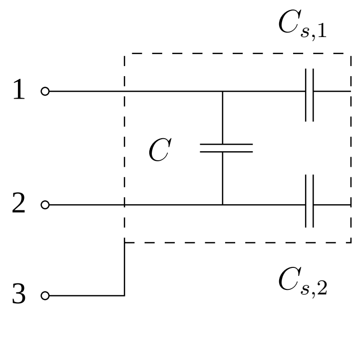
$$R_x = \frac{C_4 R_3}{C_N} = 100.9\ \Omega, \quad C_x = \frac{C_N}{G_4 R_3} = 110.0\ \text{nF}.$$

b) The loss factor is

$$\text{tg}\delta = \frac{P}{Q} = \frac{I^2 R_x}{I^2/\omega C_x} = \omega R_x C_x = \omega R_4 C_4 = 0.011.$$

c) During insulation test the insulator is placed between the plates of a capacitor. The better the insulation, the higher voltage is tolerated without strikeover. During measurement the equivalent RC model is measured on higher and higher voltage. If the system is linear the values R_x and C_x are voltage independent, but approaching the breakdown or strikeover voltage the loss factor starts to increase. This is the so-called elbow voltage. Based on the value of the elbow voltage the breakdown voltage can be estimated without damaging the device.

7.13. The model of a capacitor mounted in a metal box can be seen below:



where C is the measured capacitor, while $C_{s,1}$ and $C_{s,2}$ are the stray capacitances.

a) The series connection of the stray capacitances ($C_{s,1} = C_{s,2} = C_s = 100$ pF) is connected in parallel with the measured capacitor. The total capacitance is

$$C_e = C + \frac{C_s}{2},$$

so the error of the measurement, when measuring the capacitance between nodes '1' and '2' is

$$h = \frac{C + C_s/2 - C}{C} = \frac{C_s}{2C} = 2.5\%.$$

b) To cancel the error caused by the stray capacitors 3 wire measurement can be used. The output 'G' of the instrument has to be connected to the metal box (node '3' of the model).

c) There are three possibilities to measure the value of the stray capacitances.

1. We may use the results of the 2 and 3 wire measurements and compute

   $$C_s = 2(C_e - C).$$

   However, now we have to subtract two numbers that have almost the same value since C_s is very small in comparison with C, so the difference may have the same magnitude as that of the measurement error.

2. Using the 3 wire measurement the stray capacitances can also be measured. For example, by measuring the capacitance between nodes '1' and '3' and connecting 'G' to node '2' $C_{s,1}$ is obtained. This is more advantageous than the previous method, but in this case the current of C is canceled in the measurement which is much higher than the current of C_s, so the resistances of the wires may cause errors. Using five-wire measurement provides more accurate results.

3. The problem of "high C, low C_s" can be solved by shorting the wires of C (node '1' and '2') and measuring the capacitance between this point and node '3'. In this arrangement already the 2 wire measurement gives good results. However, the method has the following disadvantages: 1. only the value of the parallel connection of the stray capacitances can be measured and the exact value of $C_{s,1}$ and $C_{s,2}$ cannot be determined. 2. The measurement might be very sensitive to the the capacitance between the bounding box and the ground, depending on the grounding of the instrument and the stray capacitances to the ground.

Chapter 8

AD- and DA-converters

8.1. The first question to answer is how the bits at the output of the ADC represent the converted analog voltage. In other words, if the ADC is used as a voltmeter, how can the measured voltage be determined based on the digital output value. Since the converter splits the range from 0 to U_r to 2^b steps, the measured voltage U_x can be expressed as the ratio of the output value k of the ADC and its greatest possible digital output 2^b:

$$U_x = \frac{k}{2^b} U_r. \tag{8.1}$$

The error of U_r is given, 2^b is constant, and k has quantization error, since it can only be an integer. Assuming the worst case scenario, that is, the ADC is not rounding to the nearest integer, but rounds always upwards or downwards, we have

$$\Delta k = \pm 1.$$

(Note that for rounding characteristics, $\Delta k = \pm 0.5$.)

By rearranging Eq. (8.1) we get

$$k \cong \frac{U_x}{U_r} 2^b = 614.$$

After computing the sensitivities from Eq. (8.1) and using worst case summation, we obtain the expression for the absolute error as

$$\Delta U_x = \left| \frac{k}{2^b} \Delta U_r \right| + \left| \frac{U_r}{2_b} \Delta k \right| = \left| \frac{\Delta U_r}{U_r} U_x \right| + \left| \frac{U_r}{2^b} \cdot \pm 1 \right| = h_r U_x + q = 0.32 \text{ mV},$$

where $q = U_r/2^b$ is the quantization step. If $h_r = 0$, the error of the conversion depends only on the quantization step q, independently from the input voltage U_x.

The relative error is thus

$$\frac{\Delta U_x}{U_x} = \frac{\Delta U_r}{U_r} + \frac{\Delta k}{k} = h_r + \frac{q}{U_x} = 0.21\%.$$

Note that the relative error is the smallest when U_x is maximal ($U_x = U_r$), that is, when we use the full range of the converter.

8.2. The dual-slope ADC converter eliminates the effects of sinusoidal noises if the integration time is an integer multiple of the period time of the noise. If there are various periodic disturbances that should be suppressed, then the integration time should be the common multiple of the period times. The integration time that should be used Switzerland is

$$T_1 = k\frac{1}{f_1} = k \cdot 60 \text{ ms},$$

where k is an integer. In Hungary a proper value is

$$T_2 = l\frac{1}{f_2} = l \cdot 20 \text{ ms},$$

where l is again an integer. Since for $T_1 = T_2$ we have $l = 3k$, whenever k is an integer, l is integer as well. That is, the integration time we have chosen for Switzerland is perfectly fine in Hungary as well.

8.3. The dual-slope ADC computes the U_x input voltage in the following way:

$$U_x = \frac{T_x}{T}U_r, \tag{8.2}$$

where U_r is the absolute value of the reference voltage, T is the integration time, and T_x is the backward integration time.

a) Since we require $b = 20$ bits accuracy, the error of the conversion ΔU_x cannot be greater then the quantization step:

$$\Delta U_x \leq q = \frac{U_r}{2^b}.$$

Since only U_r has an error, we have

$$\Delta U_x = \frac{\partial U_x}{\partial U_r}\Delta U_r = \frac{T_x}{T}\Delta U_r,$$

and thus we need

$$\frac{T_x}{T}\Delta U_r \leq q,$$

from which

$$\Delta U_r \leq q\frac{T}{T_x} = q\frac{U_r}{U_x},$$

which has to be fulfilled for the full range of U_x input voltages. We have the most strict requirement for the error ΔU_r when the right side of the equation is the smallest, that is, when $U_x = U_r$ and $T_x = T$, thus

$$\Delta U_r \leq q = \frac{U_r}{2^b}, \quad \frac{\Delta U_r}{U_r} \leq \frac{1}{2^b} \approx 1 \text{ ppm}.$$

b) Based on the solution of problem 8.2, the integration time is the common multiple of the period of times. We can also consider the problem such that the converter has to eliminate the effects of a signal composed of two sinusoids with frequencies $f_1 = 50$ Hz and $f_2 = 60$ Hz. This means that we have to find the time period of this composite signal. The fundamental frequency is the largest common divider of the frequency components, which is $f_p = 10$ Hz in this case (see the solution of 1.6). The corresponding period time is $T_p = 1/f_p = 100$ ms. The integration time should be an integer multiple of this period time:

$$T = kT_p = k \cdot 100 \text{ ms},$$

where k is an integer.

c) Again we have to fulfill

$$\Delta U_x \leq q = \frac{U_r}{2^b}.$$

But now only T_x has an error, so we have

$$\Delta U_x = \frac{\partial U_x}{\partial T_x} \Delta T_x = \frac{U_r}{T} \Delta T_x,$$

and thus we need

$$\frac{U_r}{T} \Delta T_x \leq q,$$

from which

$$\Delta T_x \leq q \frac{T}{U_r} = \frac{U_r}{2^b} \frac{T}{U_r},$$

and so

$$\frac{\Delta T_x}{T_x} \leq \frac{T}{T_x} \frac{1}{2^b} = \frac{U_r}{U_x} \frac{1}{2^b},$$

which again has to be fulfilled for the full range of U_x input voltages. Similarly to **a)**, we have the most strict requirement for the error when $U_x = U_r$ and thus

$$\frac{\Delta T_x}{T_x} \leq \frac{1}{2^b} \approx 1 \text{ ppm}.$$

It is not surprising that we have obtained the same condition for the error of T_x as for U_r, since they are in a similar place in Eq. (8.2), that is, both are at first power.

If both errors can occur, then the sum of the two errors must be smaller then the quantization step. This way we obtain the condition

$$\left|\frac{\Delta U_r}{U_r}\right| + \left|\frac{\Delta T_x}{T_x}\right| \leq \frac{1}{2^b} \approx 1 \text{ ppm}.$$

8.4. The expression of the measured voltage is

$$U_m = U_r \frac{T_x}{T}, \tag{8.3}$$

where T is the integration time, and T_x is the backward integration time.

a) If T_x is measured by counting the clock cycles, the value of the counter is the following:

$$N = \frac{T_x}{t_0} = f_0 T_x,$$

where t_0 is the clock cycle, and f_0 is the clock frequency. The resolution of the converter is determined by the number of intervals the $[0, U_r]$ range is divided. The value of the counter for the maximal input voltage $U_x = U_r$ is

$$N_{\max} = \frac{T}{t_0} = f_0 T,$$

because in this case the integration times T and T_x equal to each other. The number of bits is thus

$$b = [\log_2(N_{\max})] = [\log_2(f_0 T)] = 18.$$

where [.] denotes the integer part operator.

b) To express the accuracy of the converter, we perform the error analysis of Eq. (8.3):

$$\Delta U_x = \left| \frac{\partial U_x}{\partial U_r} \Delta U_r \right| + \left| \frac{\partial U_x}{\partial T_x} \Delta T_x \right| + \left| \frac{\partial U_x}{\partial T} \Delta T \right|,$$

where the last term is zero, since we are counting an integer period of the clock when integrating the input voltage. Thus,

$$\Delta U_x = \left| \frac{T_x}{T} h_r U_r \right| + \left| \frac{U_r}{T} \frac{1}{f_0} \right|,$$

since the error of the backward integration time comes from the fact that we are counting integer periods, and thus the time measurement can have the maximal error of one clock period,

$$\Delta T_x = \pm t_0 = \pm \frac{1}{f_0}.$$

The error ΔU_x is maximal when $T_x = T$, thus, in the worst case we obtain

$$\Delta U_x = h_r U_r + \frac{U_r}{T} \frac{1}{f_0}.$$

The idea of determining the accuracy (or, effective number of bits b_e) is similar to problem 8.3: the error of the converter ΔU_x should be smaller than the effective quantization step $q_e = U_r / 2^{b_e}$. That is,

$$\Delta U_x = h_r U_r + \frac{U_r}{T} \frac{1}{f_0} \leq \frac{U_r}{2^{b_e}},$$

$$h_r + \frac{1}{Tf_0} \leq 2^{-b_e},$$

which is fulfilled for $b_e = 13$.

Note that while the resolution of the converter was $b = 18$ bits (we read an 18 bit number at the output), the accuracy is only $b_e = 13$ bits. This means that when using the converter as a voltmeter, the last five bits at the output of the converter are practically useless, since the number they represent is smaller than the error of the converter.

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Notations and tables

Notations

In our calculations we often use approximations. We use the notation $\approx$ if we are using a more rough approximation, e.g., $93 \approx 100$. In contrast, if we are approximating a value with a negligible error, we use the notation $\cong$, e.g., $99.999 \cong 100$.

The solutions are normally given with 4 digits, while the errors are given with 2-3 digits. This accuracy, especially for the case of the errors, is usually unnecessary, and it is only given to allow a more precise comparison of your own calculations and the solutions.

In all problems we are using the notations used in the specific area, even if this means that the same letter is used for different quantities in two problems. For example, T is used both for temperature and period time. However, within one problem the notations are unambiguous.

Table for standard normal distribution[1]

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Description: $P[0 \leq z \leq x] = p$, that is, the standard normally distributed random variable z is between 0 and x with probability p. The x value is given as a sum of the first row and first column of the table. For example, $0 \leq z \leq 1.96$ has $p = 0.475$ probability (row 1.9 and column 0.06).

[1] StatSoft, Inc. (2006). Electronic Statistics Textbook. Tulsa, OK: StatSoft. URL: `http://www.statsoft.com/textbook/stathome.html`. With the permission of StatSoft, Inc. 2300 East 14th Street, Tulsa, OK 74104, USA.

Table for Student-t distribution[2]

	p							
n	0.4	0.25	0.1	0.05	0.025	0.01	0.005	0.0005
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	31.599
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	4.141
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.768
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.646
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	3.291

Description: $P[t_n \geq x] = p$, that is, the n degrees-of-freedom t_n random variable is larger or equal than x with probability p. The first row of the table gives the p values, and below the table lists the x values for different n. For example, for a sample having $n = 20$ degrees-of-freedom, $t_n \geq 1.325$ with a probability $p = 0.1$.

[2] StatSoft, Inc. (2006). Electronic Statistics Textbook. Tulsa, OK: StatSoft. URL: `http://www.statsoft.com/textbook/stathome.html`. With the permission of StatSoft, Inc. 2300 East 14th Street, Tulsa, OK 74104, USA.