Possibilities for Modeling a Signal Structure Within a Fourier Analyzer

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***Abstract*—This paper studies a way to extend the Fourier analyzer suggested by G. Péceli, with the aim of improving the detectability of a known periodic signal component in its input signal. Possibilities for modeling a signal structure assuming the amplitude and phase relationships between its components to be known are presented.**

I. Introduction

Signal parameters may be measured and various transforms can be calculated by using recursive estimation methods based on conceptual signal models. These methods are well suited for real-time applications due to their recursive nature.

The conceptual signal model used in these methods is a hypothetical dynamical system, and the signal being measured is assumed to be its output. The state vector of the conceptual signal model, corresponding to the parameter vector of the measured signal, will then be estimated by the structure. If the conceptual signal model is deterministic, the measurement system is referred to as observer.

Hostetter introduced a recursive observer calculating the discrete Fourier transform of the input signal [1]. Péceli extended this method and suggested an observer structure consisting of signal channels containing a discrete integrator, allowing the calculation of arbitrary linear transforms of the input signal [2]. When used as a spectral observer, this structure is referred to as Fourier Analyzer (FA).

The Fourier coefficients calculated by the Fourier analyzer may be used to detect the presence of a signal component with an arbitrary spectrum within the signal being measured. In this paper, we assume a known amplitude and phase relationship between the components of the signal structure we aim to detect, and we investigate possibilities for building a corresponding model into the Fourier analyzer.

II. The Fourier Analyzer

This section presents the fundamentals of the Fourier analyzer introduced by Péceli [2].

A. The Conceptual Signal Model

The conceptual signal model used in Péceli's signal observer [2] is shown in Figure 1.

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When the observer structure is used as a spectral observer, the state variables of the conceptual signal model correspond to the complex Fourier components of the signal. Therefore, the conceptual signal model itself can be viewed as a complex multisine generator performing an inverse discrete Fourier transform (DFT) on the Fourier components of the signal. Each state variable represents a harmonic resonator of the corresponding frequency, thus these are often referred to as resonator channels.

In this paper, we will use the linear time-variant (LTV) version of the resonator-based observer as starting point. In this case, the conceptual signal model is a system described by time-variant equations, with state variables that do not vary in time: $\mathbf{x}_n$ is constant and $\mathbf{c}_n$ varies in time in (1)–(5) below. It is worth mentioning here that a linear time-invariant (LTI) realization has also been put forth [2].

The system equations describing the LTV conceptual signal model of the Fourier analyzer are as follows:

$$\mathbf{x}_{n+1} = \mathbf{x}_n, \tag{1}$$

$$\mathbf{x}_n = [x_{i,n}]^T \in \mathbb{C}^{N\times 1}, \qquad i = -K, \dots, K, \tag{2}$$

$$y_n = \mathbf{c}_n^T \mathbf{x}_n, \tag{3}$$

$$\mathbf{c}_n = [c_{i,n}]^T \in \mathbb{C}^{N\times 1}, \qquad i = -K, \dots, K, \tag{4}$$

$$c_{i,n} = e^{j2\pi i f_0 n} = z_i^n, \qquad z_i = e^{j2\pi i f_0}, \tag{5}$$

where y_n is the signal we intend to observe. The state vector $\mathbf{x}_n$ contains the $N = 2K+1$ complex Fourier components, including DC, corresponding to the K harmonics of the signal.

With a real-valued input signal, the Fourier coefficients form complex conjugate pairs: $x_{i,n} = x^*_{-i,n}$.

The relative frequency of the fundamental harmonic with respect to the sampling frequency is $f_0 = {}^{f_1}/_{f_s}$, where f_1 is the frequency of the fundamental harmonic and f_s is the sampling frequency.

The number of harmonics K is such that the following inequality holds: $K \cdot f_1 < {}^{f_s}/_2 \leq (K+1)\cdot f_1$. In case of equality, $i = -K, \dots, K+1$ and $N = 2K+2$ above.

B. The Resonator-Based Observer

An appropriately designed observer can estimate the state variables of the conceptual signal model, and thereby the complex Fourier coefficients of the input signal. Figure 2 shows the block diagram of the observer matching the LTV conceptual signal model presented in Section II-A.

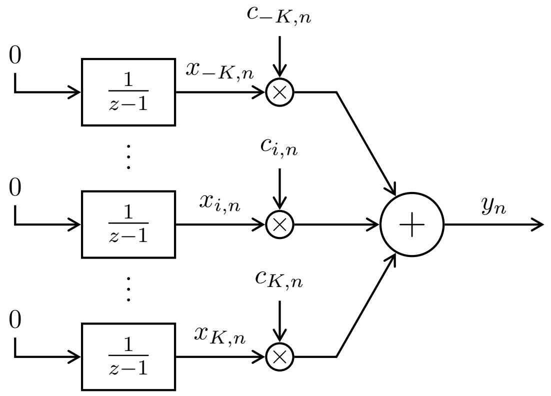

Fig. 1. Block diagram of the conceptual signal model, linear time-variant model. The integrators hold their intial preset values, corresponding to the complex Fourier coefficients of the signal. The output signal then arises as the linear combination of the integrator outputs with the time-varying $c_{i,n}$ coefficients.

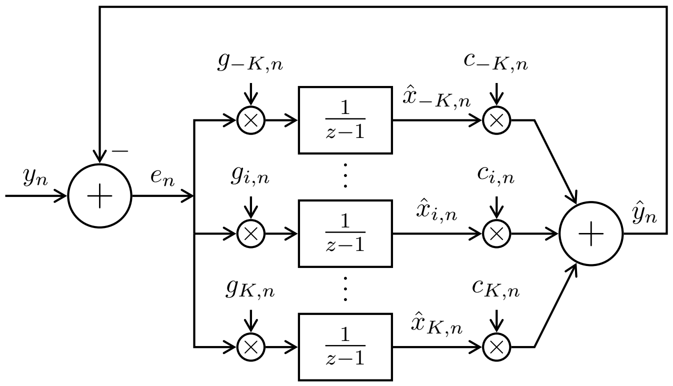

Fig. 2. Block diagram of the resonator-based observer, linear time-variant model. Notice the similarity between the structure of the observer and that of the conceptual signal model, shown in Figure 1. The observer, however, is extended by the time-varying $g_{i,n}$ coupling coefficients at the inputs of the integrators, and the common feedback.

The system equations of the observer are the following:

$$\hat{\mathbf{x}}_{n+1} = \hat{\mathbf{x}}_n + \mathbf{g}e_n = \hat{\mathbf{x}}_n + \mathbf{g}\left(y_n - \hat{y}_n\right), \tag{6}$$

$$\hat{\mathbf{x}}_n = \left[\hat{x}_{i,n}\right]^T \in \mathbb{C}^{N\times 1}, \qquad i = -K, \dots, K, \tag{7}$$

$$\hat{y}_n = \mathbf{c}_n^T \hat{\mathbf{x}}_n, \tag{8}$$

$$\mathbf{g}_n = \left[g_{i,n}\right]^T \in \mathbb{C}^{N\times 1}, \qquad i = -K, \dots, K, \tag{9}$$

$$g_{i,n} = \frac{\alpha}{N} c_{i,n}^*, \tag{10}$$

where $\hat{\mathbf{x}}_n$ is the estimated state vector, $\hat{y}_n$ is the estimate of the signal value and e_n is the error of the estimation.

The coupling vector $\mathbf{g}_n$ is the product of the observer gain α/N, a tunable parameter setting the dynamical behavior of the observer, and the coupling vector $\mathbf{c}_n$. The latter is a vector of complex roots of unity, setting the frequency response of the individual observer channels through their poles. If the coupling vectors are set according to (5) and (10), with $\alpha = 1$ and $f_1 = f_s/N$, a dead-beat observer is obtained. In this case, the transients of the observer settle in at most N steps and it produces the recursive DFT of the input signal afterwards [3].

C. On Dead-Beat Settling

The dead-beat property of the observer means the state variables of the observer converge to those of the conceptual signal model, and the error of estimation e_n becomes 0 in N (or fewer) steps. As mentioned in Section II-B, the Fourier analyzer possesses this property when the observer gain and the frequency of the fundamental harmonic are set appropriately. More generally speaking, the dead-beat nature of the observer relies on the set of the coupling vectors $\mathbf{c}_i^T$ and $\mathbf{g}_i$ constituting a biorthogonal system [2].

This corresponds to $\mathbf{C}^T = \mathbf{G}^{-1}$ with

$$\mathbf{C}^T = \begin{bmatrix} \mathbf{c}_1^T \\ \hline \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_n^T \end{bmatrix}, \; \mathbf{G} = \left[\begin{array}{c|c} \mathbf{g}_1 & \mathbf{g}_2 \ \dots \ \mathbf{g}_n \end{array}\right]. \tag{11}$$

The vectors $\mathbf{c}_i^T$ and $\mathbf{g}_i$ represent the values of the coupling coefficients corresponding to all channels at time instant i. In contrast, the column vectors of $\mathbf{C}^T$ and the row vectors of $\mathbf{G}$ contain the evolution of the coupling coefficients corresponding to a particular channel over an entire time period.

III. Modeling the Signal Structure

Let SM denote the set of positive harmonic indexes contained in the signal structure we intend to model: $SM = \{i_{s1}, i_{s2}, \dots, i_{sm}\}$. Let w_i represent the complex amplitude of the signal structure component with index i: $w_i = A_i \cdot e^{\mathrm{j}\varphi_i}$. The time function of the signal structure we are modeling can then be expressed as:

$$y_{SM,n} = \sum_{i\in SM} 2 \cdot \mathrm{Re}\left\{w_i \cdot c_{i,n}\right\}, \tag{12}$$

since the $c_{i,n}$ coefficients form complex conjugate pairs.

We need to select the dominant harmonic of the signal structure. The harmonic with the highest signal-to-noise ratio is a good candidate. Let its index be i_{s1}.

In the following, we describe the procedures we considered focusing on the harmonics with positive indexes, i.e. frequencies $0 \le f_r \le f_s/2$. Since we are assuming a real-valued input signal, all suggested modifications need to be extended to the negative counterparts of the state variables and coupling coefficients concerned, being in a complex conjugate relationship with the corresponding positive ones.

A. The Intuitive Way of Modeling

As a first attempt, we can simply bind all other harmonics of the signal structure to the dominant harmonic. That is, after the state update of the observer as in (6), we adjust the state variables belonging to the signal structure model as follows:

$$\hat{x}_{i,n+1} = \hat{x}_{i_{s1},n+1} \cdot \frac{w_i}{w_{i_{s1}}}, \quad i \in SM, \; i \ne i_{s1}. \tag{13}$$

This method, however, excludes all bound signal structure harmonics from the common feedback loop in Figure 2.

As a result, we obtain an observer that no longer provides dead-beat settling. However, reasonable tracking performance can be expected if we first let the original observer structure converge before activating the signal structure model described by (13). By design, the behavior of the signal structure model is entirely governed by the dominant harmonic.

An output waveform can be seen in Figure 3a.

B. Modifying the Basis Structure

As an attempt to improve the model, we modified the coupling vector corresponding to the dominant harmonic in such a way that it generates the whole signal structure on its own. In order to do this, we modified the appropriate column vector[1] of the coupling matrix $\mathbf{C}^T$ defined in (11):

$$\mathrm{col}_{i_{s1},\mathrm{new}}\,\mathbf{C}^T = \sum_{i \in SM} w_i \cdot \mathrm{col}_i\,\mathbf{C}^T. \tag{14}$$

The corresponding $\mathbf{g}$ coupling vectors can then be obtained as $\mathbf{G} = \left(\mathbf{C}^T\right)^{-1}$. It can be shown that the resulting values of the modified vectors can be expressed as

$$\mathrm{row}_{i_{s1},\mathrm{new}}\,\mathbf{G} = \frac{1}{w_{i_{s1}}} \cdot \mathrm{row}_{i_{s1}}\,\mathbf{G}, \tag{15}$$

$$\mathrm{row}_{i,\mathrm{new}}\,\mathbf{G} = -\frac{w_i}{w_{i_{s1}}} \cdot \mathrm{row}_{i_{s1}}\,\mathbf{G} + \mathrm{row}_i\,\mathbf{G},\ i \in SM,\ i \neq i_{s1}.$$

This yields a convergent observer with dead-beat properties by design (see Section II-C). However, as seen from (15), the behavior of the state variable carrying the signal structure model is determined by the dominant harmonic only.

An example output waveform is shown in Figure 3b.

Note. If we disable the non-dominant harmonics of the signal structure model by setting

$$\mathrm{col}_{i,\mathrm{new}}\,\mathbf{C}^T = \mathbf{0}, \quad i \in SM,\ i \neq i_{s1}, \tag{16}$$

we get a behavior matching that of the "intuitive way" described in Section III-A.

C. All Harmonics Contributing to the Signal Structure Model

By modifying the $\mathbf{C}^T$ matrix according to (14), the output of the channel carrying the modified dominant harmonic will contain all signal structure model harmonics with the amplitude and phase relations prescribed by the w_i coefficients.

We also found it desirable that all signal structure harmonics contribute to the input of this channel proportionally to their respective weights w_i. We achieved this by setting

$$\mathrm{row}_{i_{s1},\mathrm{new}}\,\mathbf{G} = \frac{1}{|SM|} \sum_{i \in SM} \frac{1}{w_i} \cdot \mathrm{row}_i\,\mathbf{G}. \tag{17}$$

By scaling with the reciprocal of the cardinality (number of elements) of the set SM, we maintain

$$\mathrm{row}_{i_{s1},\mathrm{new}}\,\mathbf{G} \cdot \mathrm{col}_{i_{s1},\mathrm{new}}\,\mathbf{C}^T = 1. \tag{18}$$

However, apart from the dominant harmonic, all signal structure harmonics are now coupled into two signal channels.

[1] The ith column and row of $\mathbf{M}$ are referred to by $\mathrm{col}_i\,\mathbf{M}$ and $\mathrm{row}_i\,\mathbf{M}$, respectively. The suffix new always indicates the new value to be assigned to the vector in question.

As a result, all harmonics of interest contribute to the signal structure model. However, dead-beat settling is not preserved and convergence becomes slower. By disabling the non-dominant signal structure harmonics according to (16), convergence can be accelerated. Since the basis vectors no longer span $\mathbb{C}^{N\times N}$, dead-beat behavior is not restored nevertheless. Figure 3c shows an output waveform.

D. A New Basis for the Subspace of the Signal Structure

We then aimed to restore the advantageous properties of the observer while keeping a single signal channel representing the entire signal structure, with all corresponding harmonics coupled onto its input. Thus we transformed the basis vectors of the subspace of $\mathbb{C}^{N\times N}$ spanned by the basis vectors contained in the signal structure model: $\left\{\mathrm{col}_i\,\mathbf{C}^T \mid i \in SM\right\}$.

We started by modifying the column corresponding to the dominant harmonic in the coupling matrix $\mathbf{C}^T$ according to (14). At this point, our goal was to transform the remaining basis vectors of the signal structure model into an orthogonal set, including the previously modified basis vector.

For the transformation, we first considered the Gram-Schmidt process [4]. Although easy to implement, the method has numerical problems, so we ended up resorting to QR decomposition by Householder reflections [5]. The leftmost columns of the resulting unitary matrix are then a set of orthonormal basis vectors spanning the subspace of $\mathbb{C}^{N\times N}$ corresponding to the signal structure model, with the first one being parallel (proportional) to the vector carrying the signal structure model we calculated earlier using (14).

Once more, the corresponding $\mathbf{g}$ coupling vectors can then be obtained as $\mathbf{G} = \left(\mathbf{C}^T\right)^{-1}$.

This yields an observer with attractive properties by design:

- The observer is convergent with dead-beat settling.
- All harmonics involved are proportionally represented in the channel carrying the signal structure model.

It still needs to be determined how the values of the other channels resulting from the QR decomposition relate to the discrepancy between the model and the actual signal.

An output waveform is shown in Figure 3d.

IV. Conclusions

We suggested several ways to model a signal structure within a Fourier analyzer. More detailed analysis will be required to ascertain whether these methods offer any actual advantage in signal detection applications.

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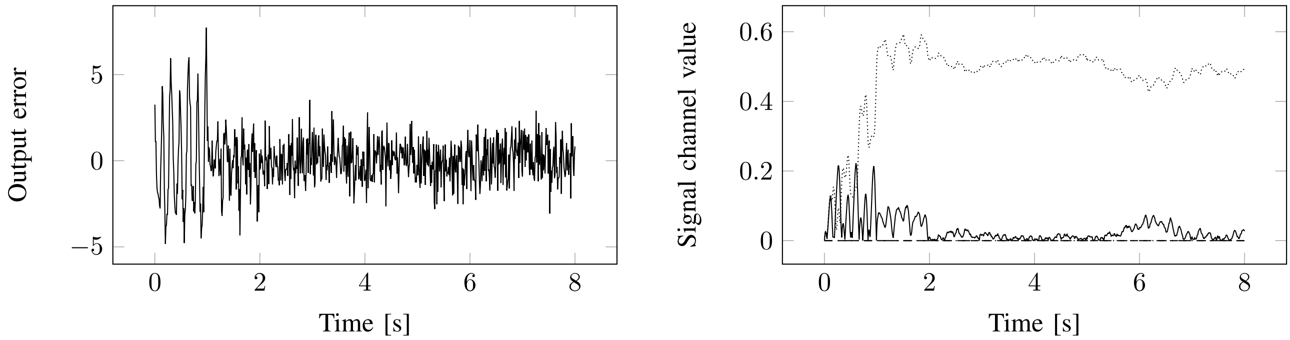
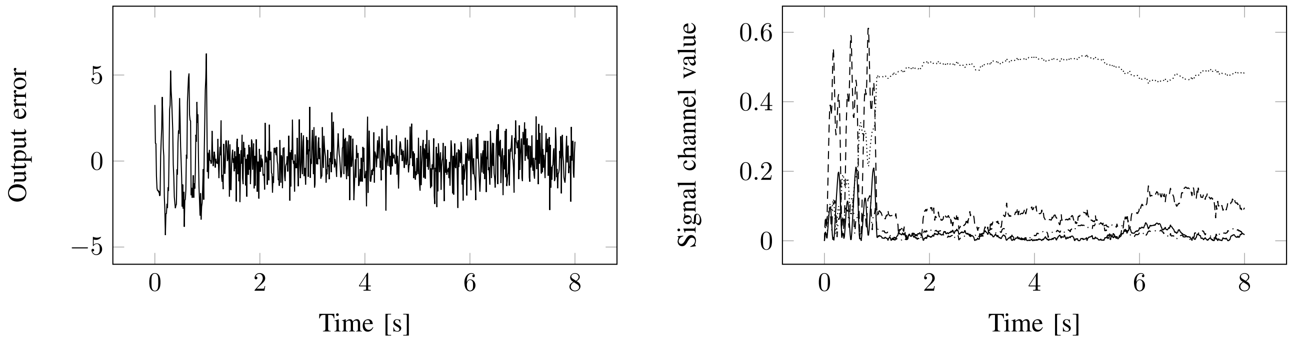
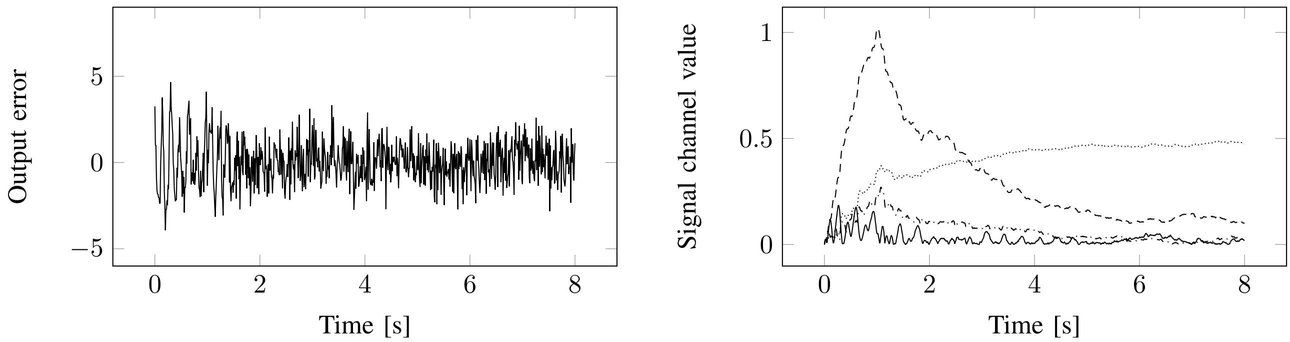


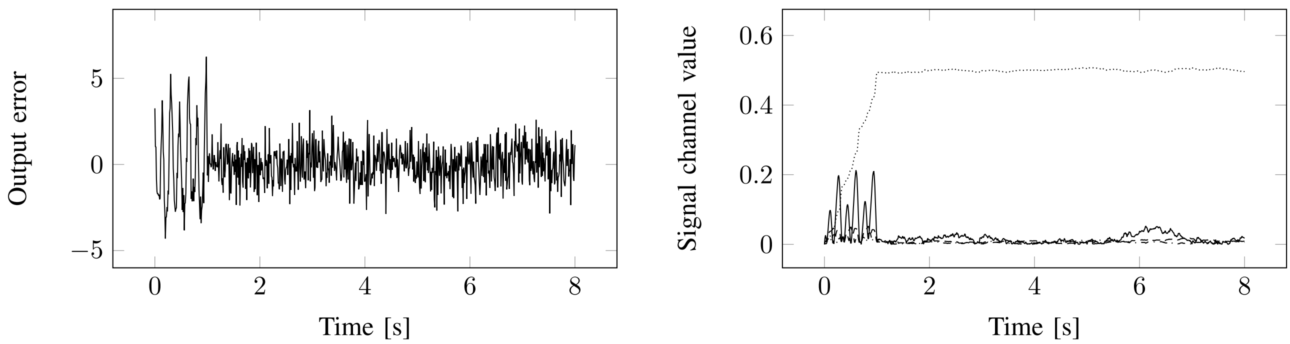
(a) The "intuitive way". The observer converges in this case, but dead-beat settling is not ensured.



(b) The dead-beat observer obtained through the first attempt at modifying the basis vectors.



(c) The signal structure model is incorporated into both the corresponding $\mathbf{g}$ and $\mathbf{c}^T$ vectors. The result is slower convergence.



(d) The dead-beat observer obtained by tayloring the basis vectors using QR decomposition.

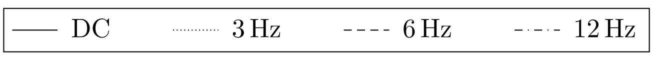


Fig. 3. Output error and signal channel values obtained with different signal structure model realizations. The parameters of the signal structure model match those of the test signal: $y(t) = \cos(2\pi \cdot f_d t) + 3 \cdot \cos(2\pi \cdot 2 f_d t + 1) + 0.7 \cdot \cos(2\pi \cdot 4 f_d t + 2)$. The test signal is contaminated with white noise, $\mathrm{SNR} = 10\,\mathrm{dB}$. For illustrative purposes, the dominant harmonic was not selected following the guideline in Section III: $f_d = 3\,\mathrm{Hz}$. In certain cases, this results in reduced performance of the signal structure model.