Uncertainty of Signal Reconstruction in the Case of Jittery and Noisy Measurements

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***Abstract*— Time domain measurements are distorted by the measurement system if the bandwidth of the system is not sufficiently high compared to that of the signal to be measured. If the distortion is known the measured signal can be compensated for it (inverse filtering or deconvolution). Since the measurement is always corrupted by noise, the reconstruction is an estimation task, i.e., the reconstructed signal may vary depending on the actual noise record.**

Our aim is to investigate the errors related to the signal reconstruction, and to provide an error bound around the reconstructed time domain waveform. Based on their nature we can distinguish between systematic and stochastic errors. In this paper, we investigate the stochastic type of errors and suggest a method to calculate the uncertainty (variance) of the reconstruction.

We developed a method for the calibration of high-speed sampling systems. Both stationary and jitter noises will be investigated.

***Index Terms*— Calibration, deconvolution, ill-posed problem, inverse problems, jitter.**

I. Introduction

MEASUREMENT of time-domain signals becomes difficult if the bandwidth of the measurement system is not high enough compared to that of the signal to be measured. In this case, the measurement system distorts the waveform. Assuming a linear and time-shift invariant model for the measurement system, the relation between the input and the output of the system can be described by convolution of the excitation signal with the impulse response of the system. If the distortion is known (impulse response or transfer function is known), the measured signal can be compensated for it. This operation is called deconvolution or inverse filtering.

Deconvolution is usually an ill-posed problem, i.e., small changes in the measured signal due to the noise cause large deviations in the reconstruction [1], [2]. To make the problem well posed, one has to define additional constraints to put a limit to the noise amplification. Several approaches are proposed in the literature [3]–[9]. The inverse filter usually suppresses those frequency bands where the noise dominates. In practical cases this means that the high-frequency noise is suppressed in different ways depending on the inverse filtering method.

Since the measurement is corrupted by noise, the reconstruction is an estimation task. The estimated input signal of the measurement system consists systematic and stochastic type of errors.

The inverse filter is not the true inverse of the measurement system, because the noise has to be suppressed in the stopband of the measurement system. This filtering distorts the signal to be measured and causes the bias of the estimate.

Variance is due to the fact that the reconstructed signal is a filtered version of the noisy measurement. The reconstructed signal can change from measurement to measurement even using the same inverse filter, since the noise records are different. We will call this uncertainty of the reconstruction, and we will distinguish it from the systematic error. With this approach the uncertainty is the fluctuation of the estimate due to the difference in successive noise records.

Our aim is to provide an error bound around the reconstructed time domain signal which shows the limitations of the reconstruction. Estimate of the systematic errors will be developed and reported in the future. The specialty of the proposed method is that it takes both stationary and nonstationary noises into account.

Our special interest is to provide uncertainty analysis for calibration of high-speed sampling systems [13]. These fast pulse oscilloscope systems work in equivalent time sampling mode with an equivalent sampling frequency of around 500 GHz. The effect of the uncertainty of the time base generator (jitter) can be modeled as a nonstationary additive noise. Quantization error and other disturbances can be modeled as stationary noises.

II. Uncertainty Analysis of the Deconvolution Result

Fast pulse sampling systems have two kinds of noise sources: 1) stationary and 2) nonstationary (jitter-related) noises.

Jitter is due to the uncertainty of the time base [11], [12]. The sample is taken at an uncertain time instant around the nominal time instant. The effect of the jitter can be modeled as an additive noise. However, the noise depends not only on the distribution of the jitter, but also on the measured signal. A good model for the relationship is to assume that the standard deviation of the jitter-related noise is proportional to the derivative of the signal [11]. The mean value of the jitter-related noise is not zero, even if the probability density function (pdf) of the uncertainty of the time base

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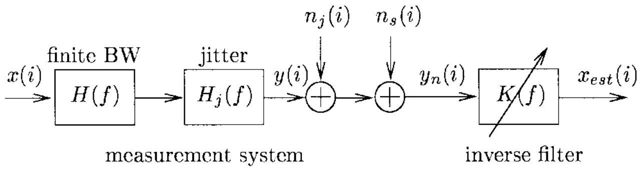

Fig. 1. Model of the measurement and inverse filtering process. $x(i)$ is the excitation signal, $y(i)$ is the system response, $n_j(i)$ is the zero mean jitter-related noise, $n_s(i)$ is the stationary noise, $y_n(i)$ is the noisy output signal, $x_{\text{est}}(i)$ is the estimated (reconstructed) input signal, $H(f)$ is the transfer function of the system, $H_j(f)$ denotes the distortion caused by the jitter, and $K(f)$ denotes the transfer function of the inverse filter.

is symmetric. The effect of the mean value of the jitter-related noise can be modeled as a filtering of the signal with the pdf of the jitter. This causes a bias in the estimate. After removing the mean value the jitter-related noise can be modeled as a nonstationary zero mean additive white noise. This zero mean noise causes fluctuation in the measurement and its effect in the reconstruction will be considered as jitter-related uncertainty. All other noise sources can be modeled as stationary additive output noises (Fig. 1).

The estimate of the signal can be written as the sum of the filtered input signal and filtered noises

$$X_{\text{est}}(k) = X(k)\underbrace{H(k)H_j(k)K(k)}_{\text{bias}} + \underbrace{N_j(k)K(k) + N_s(k)K(k)}_{\text{noise}} \quad (1)$$

where

- $X(k)$ the DFT of the input signal,
- $H(k)$ the discrete transfer function of the measurement system,
- $H_j(k)$ the DFT of the jitter pdf,
- $K(k)$ the transfer function of the inverse filter,
- $N_j(k)$ the DFT of the records of the zero mean jitter-related noises,
- $N_s(k)$ the DFT of the records of the stationary noises,
- $X_{\text{est}}(k)$ the DFT of the estimated (reconstructed) input signal.

The difference between the input signal and its estimate is

$$\begin{aligned} X(k) - X_{\text{est}}(k) &= X(k) - X(k)H(k)H_j(k)K(k) \\ &\quad - N_j(k)K(k) - N_s(k)K(k) \\ &= X(k)(1 - H(k)H_j(k)K(k)) \\ &\quad - N_j(k)K(k) - N_s(k)K(k). \end{aligned} \quad (2)$$

The effect of the jitter is usually reduced by means of signal enhancement, which reduces the variance of the noises. The correct value of the input signal expressed with the estimated value is than

$$X(k) = X_{\text{est}}(k) + \underbrace{X(k)(1 - H(k)H_j(k)K(k))}_{\text{bias}} \cdot \underbrace{-N_j'(k)K(k) - N_s'(k)K(k)}_{\text{noise}}. \quad (3)$$

The DFT of the output noise and zero mean jitter after signal enhancement are denoted by $N_j'(k)$ and $N_s'(k)$, respectively.

The error analysis provides the uncertainty bound for the stochastic part of the estimate. First the variance of the noises will be measured at a certain time instance. Than the variance of the nonstationary jitter related noise will be extrapolated for the whole time record. Next the variances will be propagated to the output of the inverse filter. After inverse filtering the distribution of the noises will converge to Gaussian one. Based on the required confidence level different bounds can be provided for the uncertainty (e.g., $2\sigma, 3\sigma$ confidence intervals).

A. Measurement of the Noise Levels

The variance of the stationary noise is constant along the time record. The jitter-related noise is nonstationary. Several measurements are averaged to reduce the effect of the jitter. Two separate records are collected during the acquisition of the data to measure the noise levels. The first one contains samples from consecutive measurements at steady topline, the other one at the sharp edge of the step-like waveform. The standard deviation of the topline will be an estimate of the level of the stationary part of the output noise, since this part of the signal does not contain jitter-related noise

$$std\_n_s = \text{std}\{y_{n,k}(t_{\text{topline}})\} \quad (4)$$

where $y_{n,k}(i)$ denotes the kth record of subsequent measurements of y_n-s, and t_{topline} denotes a time instant at the flat part of the topline.

The other record, collected at the sharp edge, contains both jitter related and output noises

$$std\_n_o(t_{\text{edge}}) = \text{std}\{y_{n,k}(t_{\text{edge}})\} \quad (5)$$

where $std\_n_o(i)$ denotes the standard deviation of the total output noise at time instant i, and t_{edge} denotes a time instant at the sharp edge. The total output noise is the sum of the stationary and jitter-related noises. The variance of the jitter-related noise can be obtained by subtracting the variance of the stationary noise from the variance of this record, since the two noise sources are independent on each other

$$std\_n_{\text{jit}}(t_{\text{edge}}) = \sqrt{std\_n_o(t_{\text{edge}})^2 - std\_n_s^2}. \quad (6)$$

If the effect of the jitter is small compared to the stationary noise, the measured level of the total output noise may be lower than the level of the stationary noise, since the measurement of the noise levels have uncertainty. In this case the effect of the jitter will not be taken into account

$$std\_n_{\text{jit}}(t_{\text{edge}}) = 0 \quad \text{if} \quad std\_n_o(t_{\text{edge}}) < std\_n_s. \quad (7)$$

B. Extrapolation of the Variance of the Jitter-Related Noise

The nonstationarity of the jitter-related noise can be well modeled by relating the variance to the derivative of the output signal

$$std\_n_{\text{jit}}(i) \sim \text{diff}\{y(i)\} \quad (8)$$

where $std\_n_{\text{jit}}(i)$ denotes the jitter noise and $\text{diff}\{y(i)\}$ denotes the derivative of the output signal. It will be assumed that the noisy observation is close enough to the noiseless output signal ($y(i)$ in Fig. 1), and the derivative of the output signal will be estimated by the derivative of the noisy observation. The level of the estimate of the standard deviation of the jitter-related noise is adjusted to the measured noise level at the edge of the signal

$$std\_n_{\rm jit}(i) \simeq {\rm diff}\{y_n(i)\} \frac{std\_n_{\rm jit}(t_{\rm edge})}{[{\rm diff}\{y_n(i)\}]_{t_{\rm edge}}} \tag{9}$$

where $std\_n_{\rm jit}(i)$ denotes the standard deviation of the jitter noise. The central difference is calculated to avoid time shift.

C. Propagation of the Variances

The noise is filtered by the inverse filter, which modifies the variance. The standard deviation of the filtered stationary noise becomes

$$std\_n_{s,\rm invfilt} = std\_n_s \sqrt{\frac{1}{Nf} \sum_{k=0}^{Nf-1} |K(k)|^2}. \tag{10}$$

The jitter-related noise is redistributed by the inverse filter, either

$$n_{\rm jit,invfilt}(i) = \sum_{l=0}^{N-1} k(l) n_{\rm jit}(i-l) \tag{11}$$

where $k(i)$ denotes the discrete impulse response of the inverse filter. The convolution becomes circular if computed by DFT. The distribution of the variance can be calculated based on the following relations:

$$\begin{aligned} {\rm var}\,\{a\eta + b\} &= a^2\,{\rm var}\,\{\eta\} & (12)\\ E\{a\eta + b\} &= aE\{\eta\} + b & (13)\end{aligned}$$

where E denotes expected value. If ζ and η are independent random variables, their variance becomes:

$${\rm var}\,\{\zeta + \eta\} = {\rm var}\,\{\zeta\} + {\rm var}\,\{\eta\}. \tag{14}$$

The variance of (11) is the convolution of the variance of the jitter noise and the squared impulse response of the inverse filter

$$\begin{aligned} {\rm var}\{n_{\rm jit,invfilt}(i)\} &= {\rm var}\left(\sum_{l=0}^{N-1} k(l) n_{\rm jit}(i-l)\right)\\ &= \sum_{l=0}^{N-1} {\rm var}\,\{k(l) n_{\rm jit}(i-l)\}\\ &= \sum_{l=0}^{N-1} k^2(l)\,{\rm var}\,\{n_{\rm jit}(i-l)\}\\ &= {\rm conv}\,(k^2(i), {\rm var}\,\{n_{\rm jit}(i)\}) \qquad (15)\end{aligned}$$

where ${\rm conv}$ denotes convolution. This convolution is circular again if the estimated input signal is calculated by means of DFT. It should be emphasized that the variance of the jitter-related noise is a function of time, since this noise is nonstationary.

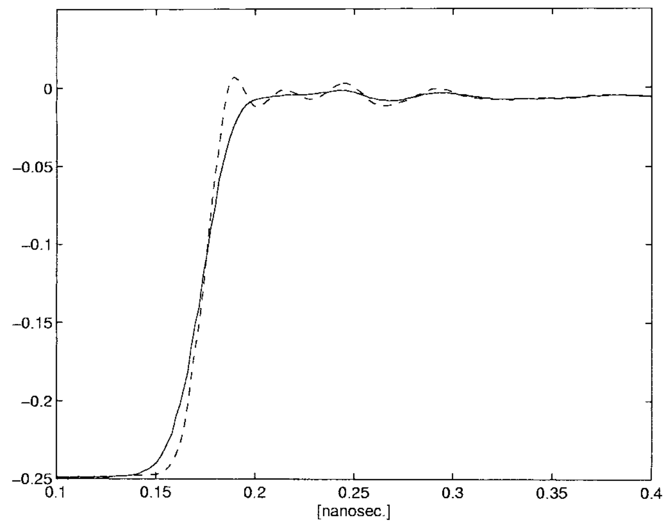

Fig. 2. Front part of simulated steplike waveform. Jitter free input signal: dashed line, noisy output signal: solid line. SNR's = 55 dB, $std_{\rm jitter} = 3\Delta t$.

D. Uncertainty Envelope

The uncertainty envelope will be based on the estimated standard deviations of the inverse filtered noises

$$\begin{aligned} uncert\_n_{s,\rm invfilt}(i) &= a\, std\_n_{s,\rm invfilt}\\ uncert\_n_{\rm jit,invfilt}(i) &= a\, std\_n_{\rm jit,invfilt}(i) \qquad (16)\end{aligned}$$

where a is a multiplication factor, expressing the level of the confidence of the uncertainty. The confidence level can be calculated from the Gaussian probability distribution

$$\begin{aligned} \Phi(a) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-(t^2/2)}\, dt\\ confidence\_level &= \Phi(a) \rightarrow a. \qquad (17)\end{aligned}$$

Since the two noise sources in (16) are independent of each other, their variances will be summed, thus the total uncertainty becomes

$$\begin{aligned} &uncert\_x_{\rm est}(i)\\ &\quad = \sqrt{uncert\_n_{s,\rm invfilt}(i)^2 + uncert\_n_{\rm jit,invfilt}(i)^2}. \end{aligned} \tag{18}$$

III. Simulations

The sampling heads of fast digital oscilloscopes are calibrated with step-like signals. We will check the results on simulated step-like waveforms. The signals are generated from measured data by filtering it to reduce the noise. Thus the dominant noise will be that added by the simulation. Nahman–Guillaume technique is used to mirror the signal and force it to be duration limited [9]. This signal is distorted by the estimated impulse response of the sampler. Jitter is simulated by distorting the waveform with the pdf of the jitter as an impulse response and adding noise to the output. Gaussian noise with standard deviation of 1 was multiplied point-by-point by the derivative of the (noiseless) output signal to simulate the nonstationary jitter noise. Stationary Gaussian output noise is added to the output, either, to simulate all other

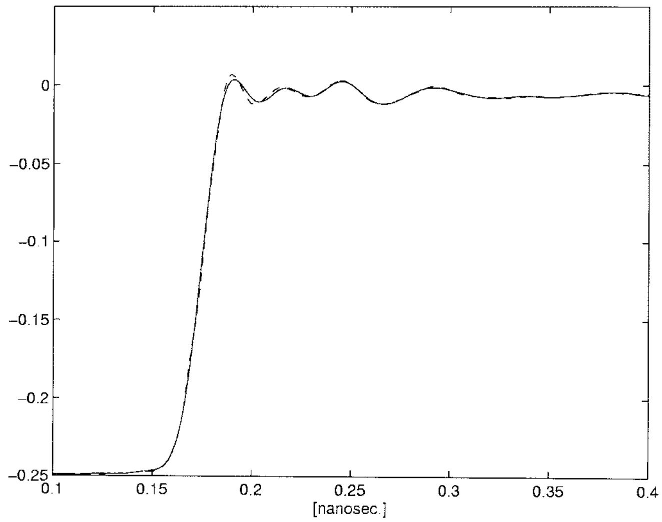

Fig. 3. Input signal (dashed line) and the estimated input signal (solid line).

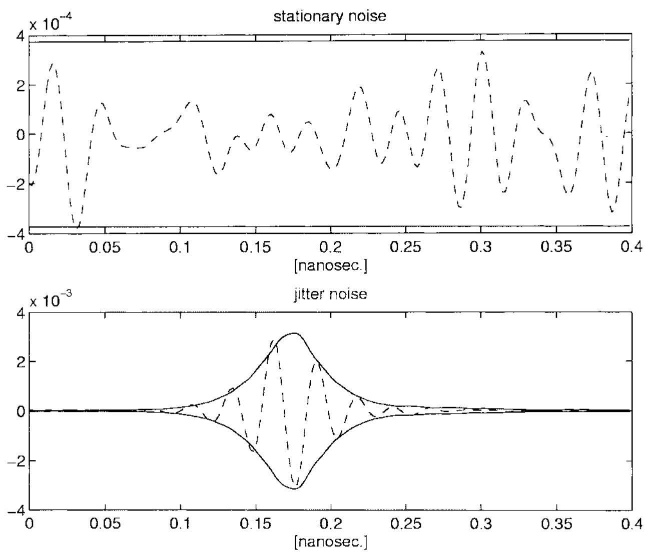

Fig. 4. Inverse filtered noises (dashed) and the estimated uncertainty bounds (solid). (2σ confidence interval) SNR $= 55$ dB, $std_{\rm jitter} = 3\Delta t$.

noise sources. The standard deviation of the jitter pdf is chosen to be $3\Delta t$, where Δt is the sampling interval. The signal-to-noise ratio (SNR) of the stationary noise is set to 55 dB. The signal to be measured and the distorted noisy one are depicted in Fig. 2. (Only the front parts of the signals will be shown to magnify the transition part.)

We used the regularization method to reconstruct the input waveform [9], [10] (Fig. 3). The uncertainty of the reconstruction is calculated based on the equations described in Section II. The resulting bounds (2σ bound) and the simulated inverse filtered noises are compared in Fig. 4. The total uncertainty is obtained by adding the effects of jitter-related and stationary noises (Fig. 5). The uncertainty envelope around the reconstructed signal is shown in Fig. 6.

For 2σ confidence level the fluctuation of the signal should remain within the estimated bounds 95.45% of the time. We calculated the statistics of the "in-bound time" for 1000

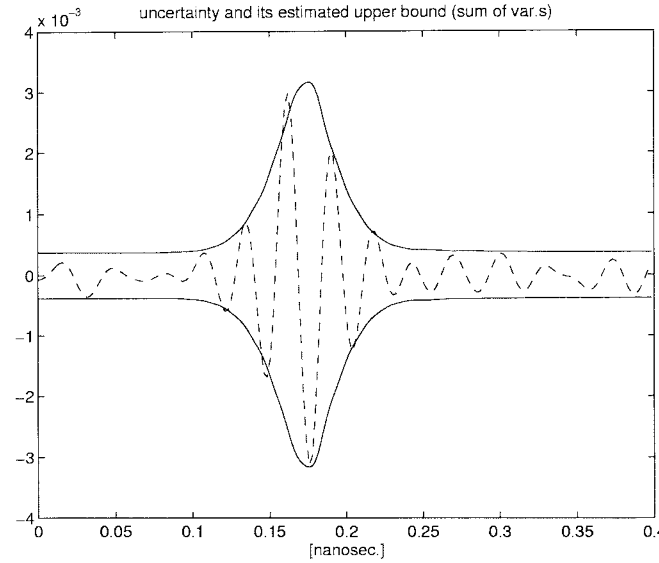

Fig. 5. Estimated total uncertainty bound (solid) and the level of filtered noises (dashed). SNR $= 55$ dB, $std_{\rm jitter} = 3\Delta t$.

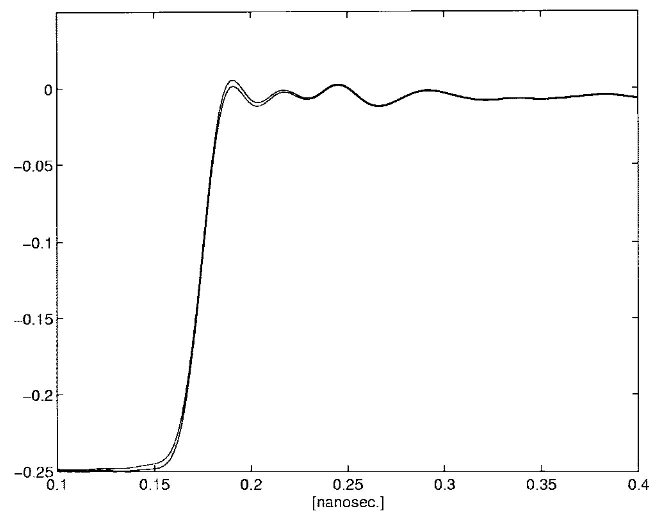

Fig. 6. Uncertainty envelope around the reconstructed signal.

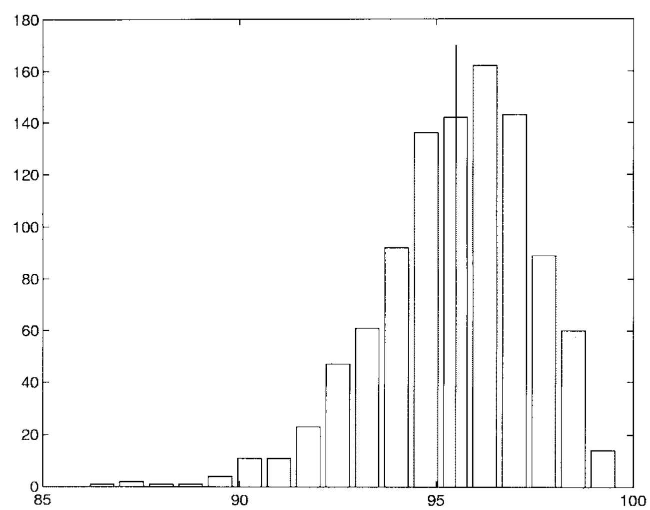

Fig. 7. Histogram of the "in-bound time" for 1000 noise records, and the mean value (95.5%).

successive noise records (Fig. 7). The mean value of the distribution is 95.5%, which is very close to the expectation.

IV. Conclusions

Uncertainty analysis of signal reconstruction has been carried out. We modeled jitter-related nonstationary noise and stationary noises from different sources (e.g., quantization, electromagnetic interferences). We developed a method to calculate uncertainty bound around the reconstructed waveform, based on the required confidence level. The uncertainty bound is a function of time because of the nonstationary nature of the jitter-related noise. We aim at developing bias-related bounds in the future to extend the error analysis.

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