SAT/SMT/AR
Introduction and Applications

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About me

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  - Supervisor: Dr. Zoltán Micskei
  - Topic: Applying counterexample-guided abstraction refinement in model checking

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References

- This presentation is based on the talks presented at the SAT/SMT/AR Summer School 2016

- Some content (text, figures, examples) of this presentation is copied from the presentations of the school, available at http://ssa-school-2016.it.uu.se/programme/

History of the school

- SAT/SMT (2011 – 2015)
  - 2011: MIT, USA
  - 2012: Trento, Italy
  - 2013: Espoo, Finland
  - 2014: Semmering, Austria
  - 2015: Stanford, USA

- SAT/SMT/AR (2016 –)
  - 2016: Lisbon, Portugal
    - http://ssa-school-2016.it.uu.se/

Overview

- SAT
  - Boolean satisfiability
  - Boolean variables and operators
  - Problem: is the formula satisfiable?

$$\neg p \wedge (p \vee q)$$

- AR
  - Automated Reasoning
  - First order logic
  - Problem: theorem proving

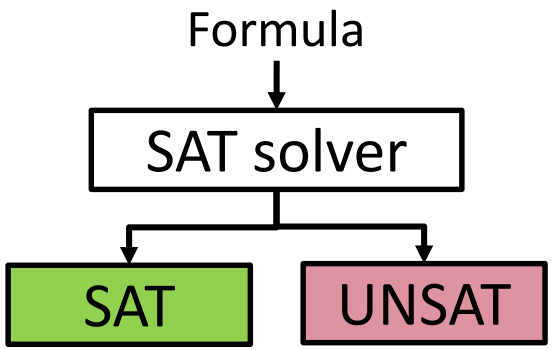
$$\forall x, y\ \exists z: p(f(x, y), g(z))$$

- SMT
  - Satisfiability Modulo Theories
  - First order logic, but interpreted symbols
  - Problem: is the formula satisfiable?

$$(x \leq y + 1) \wedge (y \geq 3)$$

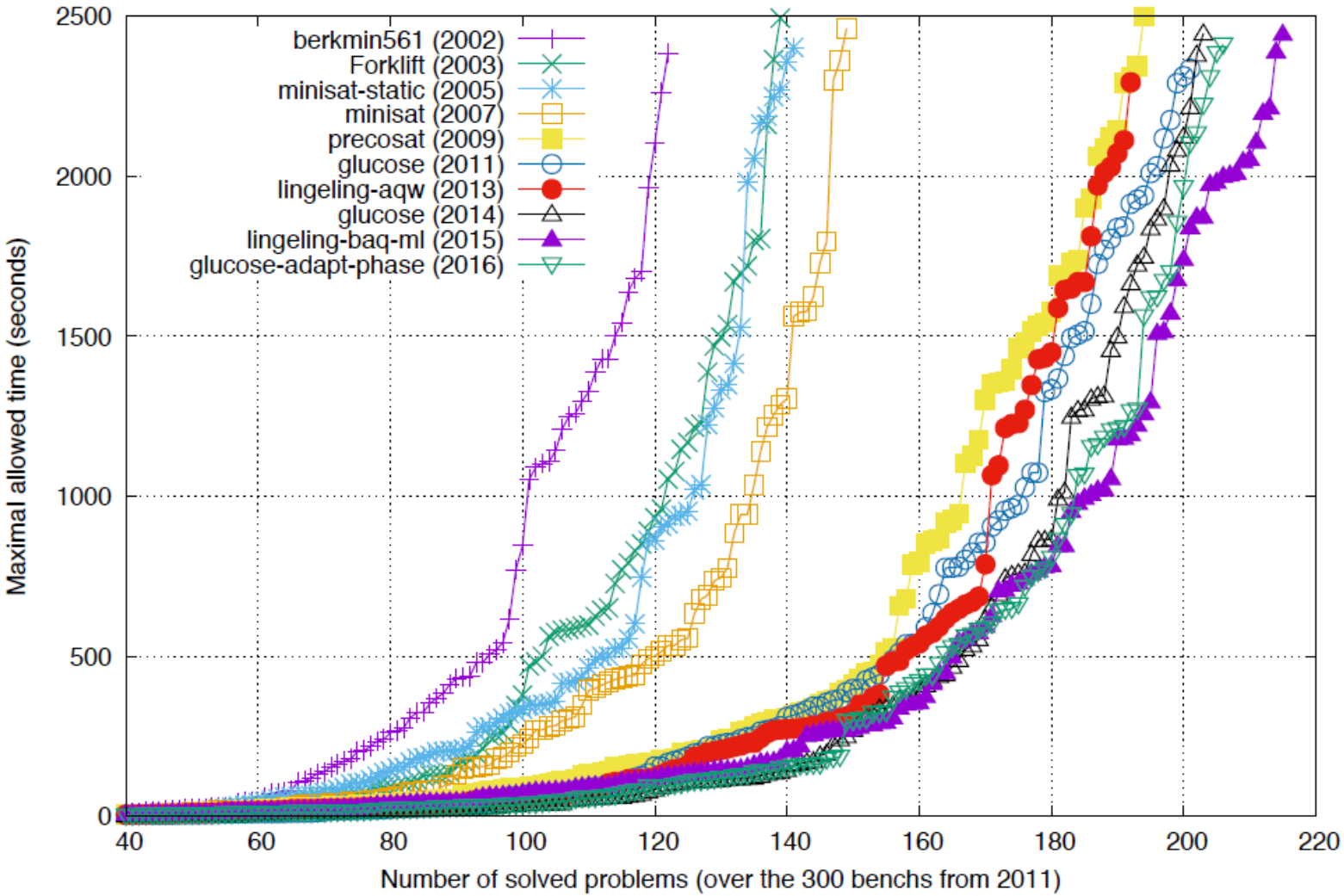
SAT basics

- Building blocks
  - Boolean variables $x, y, z, \ldots$
  - Literals (variable or negation) $x, \neg x, y, \neg y, z, \neg z, \ldots$
  - Clauses (disjunction of literals) $(x \vee y \vee \neg z)$
  - Formulas (conjunction of clauses) $(x \vee y) \wedge (\neg x \vee z)$
- SAT problem
  - Decide if a variable assignment exists that evaluates the formula to true
  - In theory: NP-complete
  - In practice: efficient solvers



→Annual SAT competition: http://baldur.iti.kit.edu/sat-competition-2016/

SAT in practice



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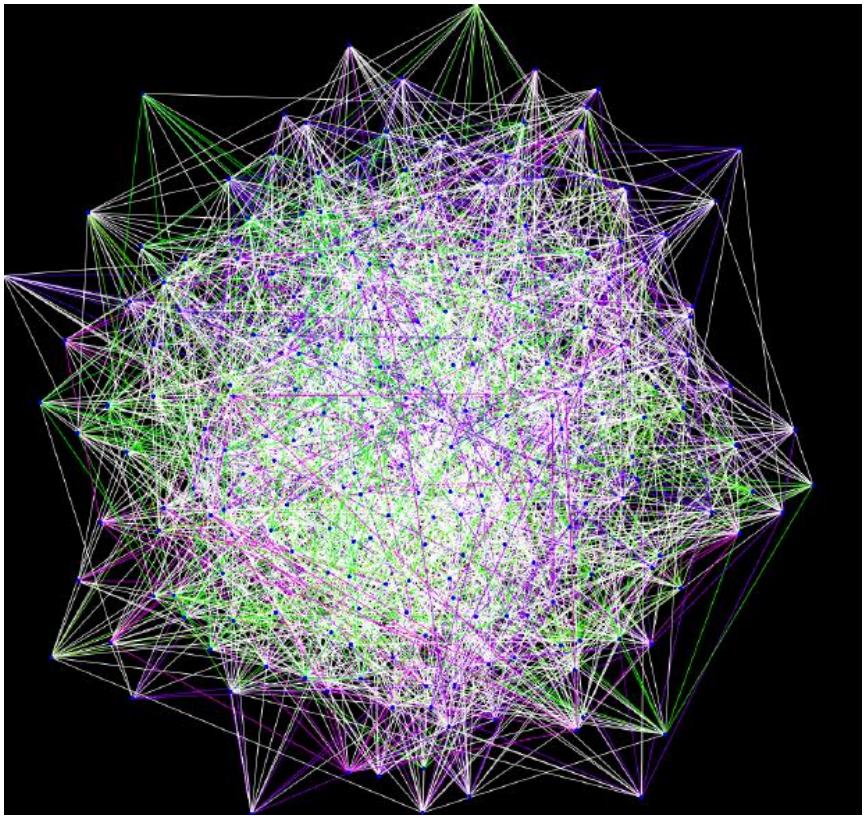
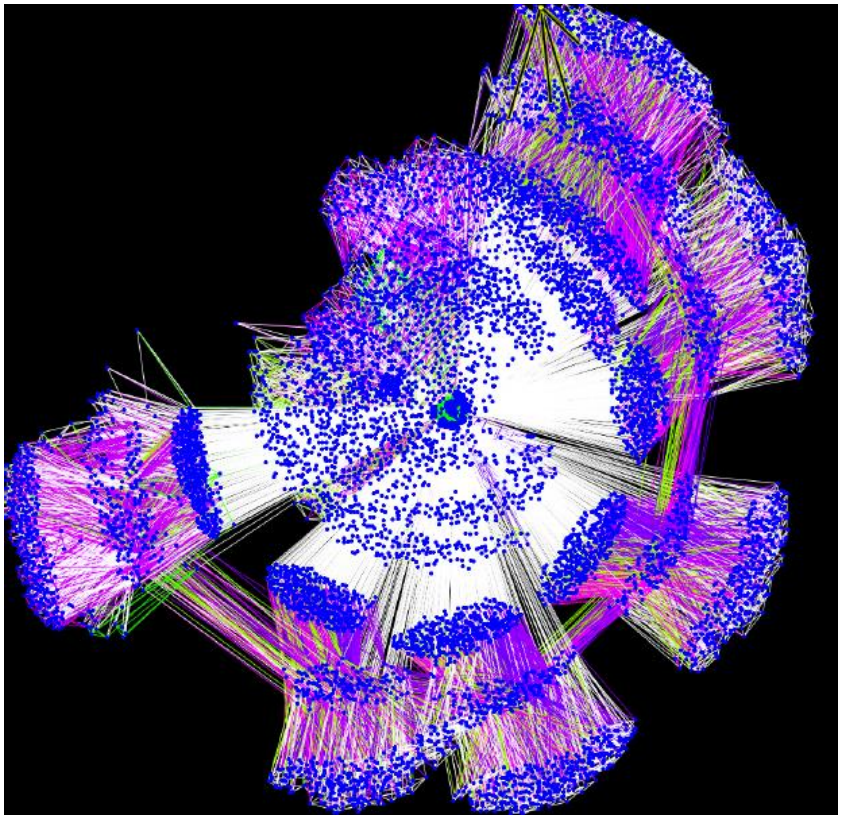
SAT in practice

Randomly generated problem



Real world problem (bounded model checking)



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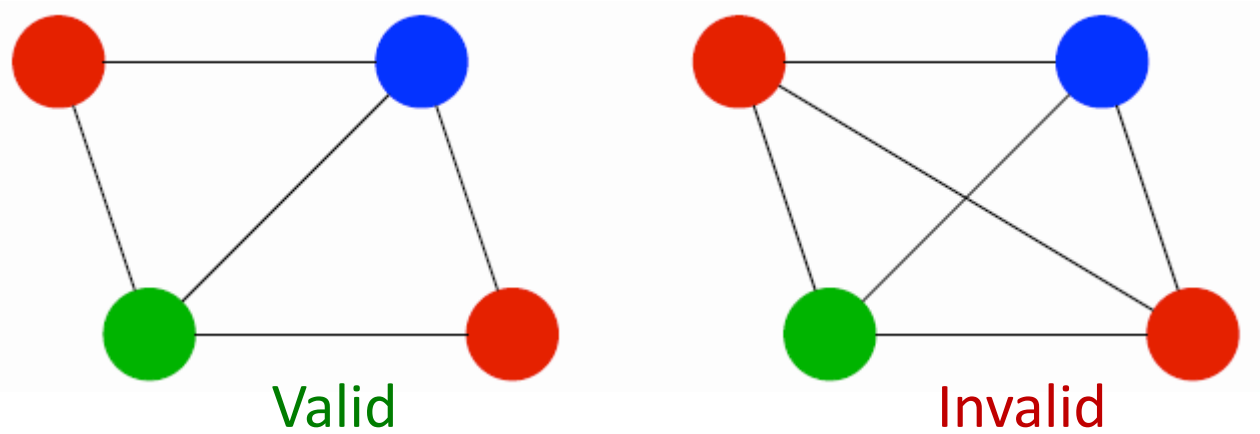
SAT in practice

- Wide range of applications

http://ssa-school-2016.it.uu.se/wp-content/uploads/2016/06/jpms-satsmtar16-slides.pdf

SAT modeling

- Graph $G = (V, E)$, coloring with k colors
  - $x_{i,j} = 1$ if vertex v_i has jth color
  - Guarantee valid coloring
    - $(\neg x_{i,j} \vee \neg x_{l,j})$ for all $(v_i, v_l) \in E$ and $j \in \{1, \dots, k\}$
  - All vertices should have some color
    - $x_{i,1} \vee x_{i,2} \vee \cdots \vee x_{i,k}$ for all $v_i \in V$



SAT modeling

- Sudoku
  - $v_{i,j,k} = 1$ if the value in row i and cell j is k
  - Each cell has exactly one value
    - $\sum_{k=1}^{9} v_{i,j,k} = 1$ for all $i, j \in \{1, \dots, 9\}$
  - Each value used exactly once in each row
    - $\sum_{j=1}^{9} v_{i,j,k} = 1$ for all $i, k \in \{1, \dots, 9\}$
  - Each value used exactly once in each column
    - $\sum_{i=1}^{9} v_{i,j,k} = 1$ for all $j, k \in \{1, \dots, 9\}$
  - Each value used exactly once in each 3x3 sub-grid
    - $\sum_{r=1}^{3} \sum_{s=1}^{3} v_{3i+r,3j+s,k} = 1$ for all $i, j \in \{0, \dots, 2\}$ and $k \in \{1, \dots, 9\}$
  - Fixed cells
    - $v_{1,1,5} = 1, v_{1,2,3} = 1, v_{1,5,7} = 1, v_{2,1,6} = 1, \dots$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Modeling challenges

- Non-conjunctive formulas

$$(p \vee q) \leftrightarrow (p \wedge q \vee r)$$

  - Introduce auxiliary variables (Tseitin transformation)
  - Linear increase of the formula size
- Cardinality constraints
  - $\sum_{j=1}^{n} x_j \geq 1$: $x_1 \vee x_2 \vee \cdots \vee x_n$
  - $\sum_{j=1}^{n} x_j \leq 1$:
    - Pairwise: $O(n^2)$ clauses
    - Sequential counter: $O(n)$ clauses + $O(n)$ new variables
    - Bitwise: $O(n \log n)$ clauses + $O(\log n)$ new variables
  - Can be generalized from 1 to k

Modeling challenges

- Pseudo-Boolean constraints
  - $\sum_{j=1}^{n} a_j x_j \leq b$
  - Binary Decision Diagram (BDD): exponential clauses
  - Watchdog encoding: $O(n^3 \log(n) \log(a_{max}))$ clauses + $O(n^2 \log(n) \log(a_{max}))$ new variables
- CSP constraints
  - Direct encoding
    - x_i variable with domain D_i, $m_i = |D_i|$
    - $x_{i,1}, \ldots, x_{i,m_i}$ variables, $\sum_{k=1}^{m_i} x_{i,k} = 1$
    - If $x_i = v_i \wedge x_j = v_j$ is not valid, add a clause: $(\neg x_{i,v_i} \vee \neg x_{j,v_j})$
  - There are other, more efficient encodings

SAT solver algorithms

- DP-60
  - Eliminate variables step by step with resolution

$$x_1 \vee x_4$$
$$\neg x_1 \vee x_4 \vee x_{14}$$
$$\neg x_1 \vee \neg x_3 \vee \neg x_8$$
$$x_1 \vee x_8 \vee x_{12}$$
$$x_1 \vee x_5 \vee \neg x_9$$
$$x_2 \vee x_{11}$$
$$\neg x_3 \vee \neg x_7 \vee x_{13}$$
$$\neg x_3 \vee \neg x_7 \vee \neg x_{13} \vee x_9$$
$$x_8 \vee \neg x_7 \vee \neg x_9$$

$$x_1 \vee \begin{pmatrix} x_4 \\ x_8 \vee x_{12} \\ x_5 \vee \neg x_9 \end{pmatrix}$$
$$\neg x_1 \vee \begin{pmatrix} x_4 \vee x_{14} \\ \neg x_3 \vee \neg x_8 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_8 \vee x_{12} \\ x_5 \vee \neg x_9 \end{pmatrix} \vee \begin{pmatrix} x_4 \vee x_{14} \\ \neg x_3 \vee \neg x_8 \end{pmatrix}$$

$$x_4 \vee x_4 \vee x_{14}$$
$$x_4 \vee \neg x_3 \vee \neg x_8$$
$$x_8 \vee x_{12} \vee x_4 \vee x_{14}$$
$$x_8 \vee x_{12} \vee \neg x_3 \vee \neg x_8$$
$$x_5 \vee \neg x_9 \vee x_4 \vee x_{14}$$
$$x_5 \vee \neg x_9 \vee \neg x_3 \vee \neg x_8$$

SAT solver algorithms

- DPLL-62
  - Backtrack search
    - Pick a literal
    - Search for solution by setting the literal to true
    - If conflicting, set literal to false

$x_1 \vee x_4$
$\neg x_1 \vee x_4 \vee x_{14}$
$x_1 \vee \neg x_3 \vee \neg x_8$
$x_1 \vee x_8 \vee x_{12}$
$x_2 \vee x_{12}$
$\neg x_3 \vee \neg x_{12} \vee x_{13}$
$\neg x_3 \vee x_7 \vee \neg x_{13}$
$x_8 \vee \neg x_7 \vee \neg x_{12}$

Level	Literal	Backtrack
1	$\neg x_1$	X
	x_4	
2	x_3	X
	$\neg x_8$	
	x_{12}	
	x_{13}	
	x_7	

Conflict, backtrack

SAT solver algorithms

- DPLL-62 problems
  - Selecting the literal
    - Bad decisions at the beginning can have a high cost
  - Only chronological backtracking

- CDCL algorithms (2001 –)
  - Conflict Driven Clause Learning
  - Learn new clause from conflict
  - Non-chronological backtrack and restart is possible

SAT extensions

- MUS: Minimal Unsatisfiable Subset
  - Minimal explanation of inconsistency
- MCS: Minimal Correction Subset
  - Minimal relaxation to recover consistency
- Application: analysis of over-constrained systems
  - Software/hardware model checking
  - Model-based diagnosis
  - Maximum feasible subsystem
  - Timetable/scheduling
  - …

SAT extensions

- MaxSAT
  - Maximize the number of satisfied clauses
  - Weighted MaxSAT: assign weights to clauses
  - Partial MaxSAT: „hard” clauses that must be satisfied
  - Application: optimization problems
- QBF (Quantified Boolean Formulas)
  - Quantified formulas ($\forall, \exists$)
  - Application: e.g. bounded model checking
  - Harder than SAT if NP $\neq$ PSPACE
    - This is reflected in the complexity of solvers
    - But allows for exponentially more succinct encodings than SAT

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$\forall x, y\ \exists z: p(f(x, y), g(z))$

- SMT
  - Satisfiability Modulo Theories
  - First order logic, but interpreted symbols
  - Problem: is the formula satisfiable?

$(x \leq y + 1) \wedge (y \geq 3)$

AR basics

- Expressiveness of SAT is not always enough
- First Order Logic (FOL)
  - Symbolic knowledge representation and reasoning
  - Building blocks
    - Simple terms: constants, variables $a, b, c, \dots x, y, z, \dots$
    - Complex terms: functions $f(a, b), g(z)$
    - Atoms: predicates $p(x, y), a = b$
    - Literals: atom or negation $p(x, y), \neg p(x, y)$
    - Formulas: connectives and quantifiers $\forall x \exists y: p(f(x), y) \wedge x \neq y$
    - Partial order between elements $f\big(x, g(x)\big) \succ g(x)$

AR basics

- Theorem proving
  - H set of formulas: assumptions, hypotheses
  - φ formula: conjecture
  - Theorem proving problem: $H \Rightarrow \varphi$?
  - Drawback: infinitely many interpretations on infinitely many domains
- In theory
  - Gödel: completeness
  - Turing: undecidability
  - Herbrand: semi-decidability

AR basics

- Applications
  - Software and hardware verification
  - Static analysis
  - Reasoning in knowledge bases
  - Mathematical theorem proving
- Example: arrays
  - Axioms
    - $\forall a, i, j: i = j \rightarrow a[i] = a[j]$
    - $\forall a, v, i, j: i = j \rightarrow \text{store}(a, i, v)[j] = v$
    - $\forall a, v, i, j: i \neq j \rightarrow \text{store}(a, i, v)[j] = a[j]$
  - Theorem
    - $\forall a, i, j:$
      $\text{store}(\text{store}(a, i, a[j]), j, a[i]) = \text{store}(\text{store}(a, j, a[i]), i, a[j])$

Theorem provers

- Inference system
  - Set of inference rules
  - Expansion or contraction of a set of formulas
- Expansion
  - Resolution

$$\frac{p(x) \vee q(x), \neg p(f(a)) \vee \mathrm{r(a)}}{q(f(a)) \vee \mathrm{r(a)}}$$

  - Superposition

$$\frac{f(z,e) = z, f(l(x,y),y) = x}{l(x,e) = x}$$

- Contraction
  - Subsumption

$$\frac{p(x,y) \vee q(z), p(a,a) \vee q(b) \vee \mathrm{r(a)}}{p(x,y) \vee q(z)}$$

  - Simplification

$$\frac{f(x) = x, p(f(a)) \vee \mathrm{q(a)}}{f(x) = x, p(a) \vee \mathrm{q(a)}}$$

Theorem provers

- Search plan
  - Which rule to use?
  - Fairness: use all rules that are needed
  - Redundancy: avoid redundant clauses
- Strategy = inference system + search plan
- Algorithms
  - Ordering-based
  - Subgoal-reduction
  - Semantically-Guided Goal-sensitive

AR in practice

- Tools
  - Otter, EQP, Prover9
  - SNARK
  - SPASS
  - E
  - Vampire
  - leanCoP
  - iProver
  - Meris, MetiTarski
- CASC
  - The CADE ATP System Competition

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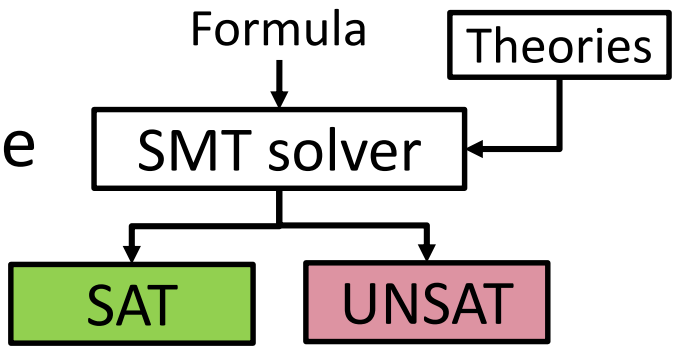
  - Problem: is the formula satisfiable?

$(x \leq y + 1) \wedge (y \geq 3)$

SMT basics

- First order logic is undecidable ☹
- In practice, symbols often have an interpretation in some theory
  - Integers, reals, arrays, …
  - Decidable theories (or fragments) ☺
- SMT problem
  - Check if a FOL formula is satisfiable modulo the background theories
  - Example (QF_UFLRA)

$$(z = 1 \vee z = 0) \wedge (x - y + z = 1) \wedge \big(f(x) > f(y)\big)$$



SMT modeling

- Example: schedule 3 jobs, each composed of 2 consecutive tasks, on 2 machines in 8 time units

	Machine1	Machine2
Job1	2	1
Job2	3	1
Job3	2	3

$$(t_{1,1} \geq 0) \wedge (t_{1,2} \geq t_{1,1} + 2) \wedge (t_{1,2} + 1 \leq 8)$$
$$(t_{2,1} \geq 0) \wedge (t_{2,2} \geq t_{2,1} + 3) \wedge (t_{2,2} + 1 \leq 8)$$
$$(t_{3,1} \geq 0) \wedge (t_{3,2} \geq t_{3,1} + 2) \wedge (t_{3,2} + 3 \leq 8)$$
$$((t_{1,1} \geq t_{2,1} + 3) \vee (t_{2,1} \geq t_{1,1} + 2)$$
$$((t_{1,1} \geq t_{3,1} + 2) \vee (t_{3,1} \geq t_{1,1} + 2)$$
$$((t_{2,1} \geq t_{3,1} + 2) \vee (t_{3,1} \geq t_{2,1} + 3)$$
$$((t_{1,2} \geq t_{2,2} + 1) \vee (t_{2,2} \geq t_{1,2} + 1)$$
$$((t_{1,2} \geq t_{3,2} + 3) \vee (t_{3,2} \geq t_{1,2} + 1)$$
$$((t_{2,2} \geq t_{3,2} + 3) \vee (t_{3,2} \geq t_{2,2} + 1)$$

$$t_{1,1} = 5 \quad t_{1,2} = 7$$
$$t_{2,1} = 2 \quad t_{2,2} = 6$$
$$t_{3,1} = 0 \quad t_{3,2} = 3$$

SMT modeling

- Example: is the code correct?

```
void swap(int* a, int* b) {
    *a = *a + *b;
    *b = *a - *b;
    *a = *a - *b;
}
```

$$h_1 = \text{store}(h_0, a, h_0[a] +_{32} h_0[b])$$
$$h_2 = \text{store}(h_1, b, h_1[a] -_{32} h_1[b])$$
$$h_3 = \text{store}(h_2, a, h_2[a] -_{32} h_2[b])$$

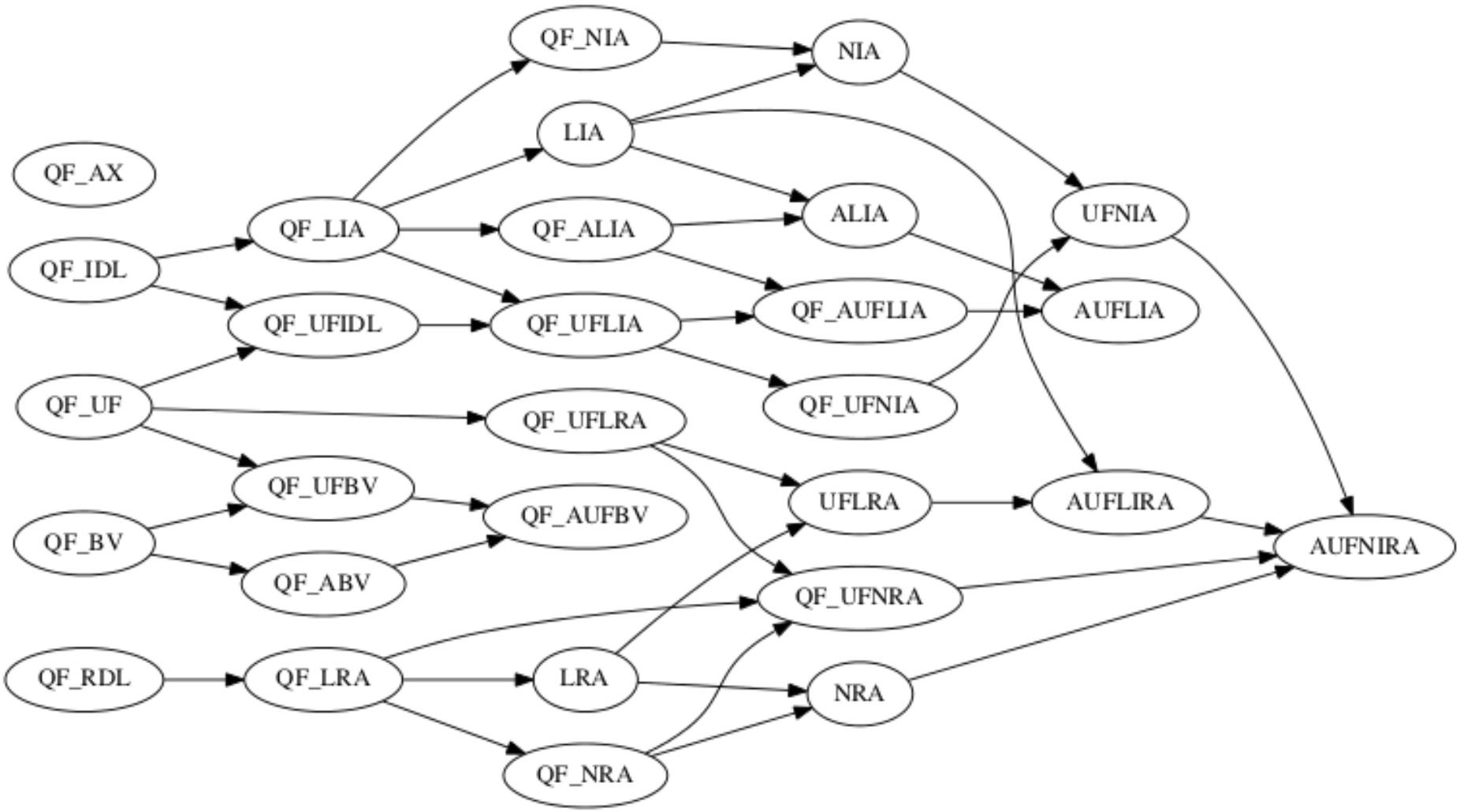
$$\neg(h_3[a] = h_0[b] \wedge h_3[b] = h_0[a])$$

$$a = 0, b = 0$$
$$h_0[0] = 1, h_1[0] = 2$$
$$h_2[0] = 0\ h_3[0] = 0$$

SMT theories

- Uninterpreted functions and equality $f(g(x)) = x$
- Arrays $\mathrm{store}(a, i, e)[i] = e$
- Arithmetic (integer/ real)
  - Difference logic $x - y \leq 1$
  - Linear arithmetic $2x - 3y + 4z \leq 5$
  - Nonlinear arithmetic $x^2 + 3xy + y^2 > 0$
- Bitvectors $(\sim a \,\&\, (a +_{32} 1)) > a$
- Quantifiers $\forall x \exists y$
- …
- Combinations are also possible

SMT logics



http://smtlib.cs.uiowa.edu/logics.shtml

SMT algorithms

- Theory solver: decision procedure for a conjunction of literals in a theory
  - Example: linear real arithmetic
    - Literals: linear (in)equalities
    - Conjunction: system of linear (in)equalities → simplex algorithm
- Basic algorithm
  - Convert formula to disjunctive normal form
    - $(\neg l_0 \wedge \neg l_1 \wedge l_2) \vee (l_3 \wedge \neg l_4) \vee (l_5 \wedge l_6)$
  - Check if any of the disjuncts is satisfiable
- Lazy algorithm
  - Introduce Boolean variables for SMT literals
  - Apply SAT solver, check literals selected by the solver
    - If not satisfiable → add a blocking clause

Lazy SMT

- Example

Theory solver

$\neg(a \geq 3) \wedge (a \geq 3 \vee a \geq 5)$

$\neg(a \geq 3) \wedge (a \geq 5)$

UNSAT

$(a \geq 3) \vee \neg(a \geq 5)$

Blocking clause

SAT solver

$\neg x \wedge (x \vee y)$

$x = 0, y = 1$

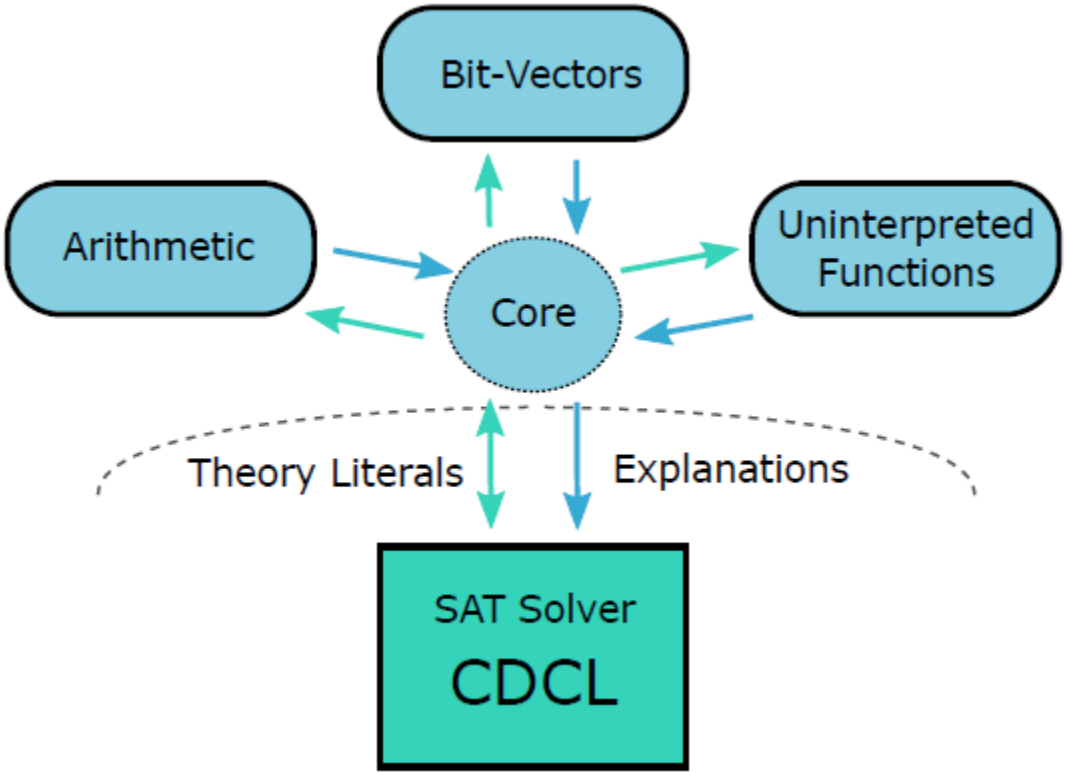
$(x \vee \neg y) \wedge \neg x \wedge (x \vee y)$

UNSAT

$x \vee \neg y$

SMT algorithms

- DPLL(T) algorithm
  - Lazy SMT + tight integration with SAT solver
    - Incremental
    - Backtrack
    - Propagation
    - Conflicts



http://ssa-school-2016.it.uu.se/wp-content/uploads/2016/06/jovanovic.pdf

Theory solvers

- Uninterpreted functions
  - Congruence closure
- Difference logic
  - Map to a graph and search for negative cycle
- Linear arithmetic
  - Reals: variant of simplex algorithm
  - Integers: branch & bound algorithm
- Arrays
  - Use uninterpreted functions and refine if needed
- Bitvectors
  - Map to SAT (bit blasting)

SMT in practice

- SMT-LIB: international initiative
  - Standard descriptions of theories
  - Common input/output language
  - Connect the community
  - Large benchmark library
  - Related software tools
- Actively developed solvers (as of 2015)
  - Alt-Ergo, AProVE, Boolector, CVC4, MathSAT 5, OpenSMT 2, raSAT, SMTInterpol, SMT-RAT, STP, veriT, Yices 2, Z3

http://smtlib.cs.uiowa.edu/

SMT in practice

```
> (set-option :print-success false)
> (set-option :produce-models true)
> (set-option :interactive-mode true)
> (set-logic QF_LIA)
> (declare-fun x () Int)
> (declare-fun y () Int)
> (assert (= (+ x (* 2 y)) 20))
> (assert (= (- x y) 2))
> (check-sat)
sat
> (get-value (x y))
((x 8)(y 6))
```

Constant: function with no parameter

$x + 2y = 20$
$x - y = 2$

$x = 8$
$y = 6$

http://smtlib.github.io/jSMTLIB/SMTLIBTutorial.pdf

Summary

- SAT

$$\neg p \wedge (p \vee q)$$

  - Success story, engine of the engines, low level
- AR

$$\forall x, y\ \exists z: p(f(x,y), g(z))$$

  - High level, undecidable in general
- SMT

$$(x \leq y + 1) \wedge (y \geq 3)$$

  - High level, but decidable theories and fragments

Questions?

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