A Comparative Analysis of Fuzzy t-Norm Approaches to the Measurement Uncertainty Evaluation

Claudio De Capua and Emilia Romeo

DIMET, University Mediterranea of Reggio Calabria
Via Graziella (Località Feo di Vito), 89100 Reggio Calabria, Italy
Phone: +39-0965-875364, Fax: +39-0965-875247
Email: decapua@unirc.it; emilia.romeo@unirc.it

Abstract. The demands of statistical investigations in measurements inspired the remarkable development of probabilistic methods [1]. However, the probability theory didn't prove to be fully adequate for all types of uncertainty. Probability theory is excellent if the ambiguity is to be modelled, but its attempts to describe vagueness is quite inconsistent with common sense [2]. Fuzzy theorists have often argued that a major motive behind the theory of fuzzy sets has been the treatment of *uncertainty*. In particular way, it's well accepted that a measurement result (no matter what kind of instruments we are using in our process) is just a number which is only known to lie within an interval, and this is the reason for which fuzzy sets can be successfully applied [3]. To consider both systematic and random effect of measurement operation, in agreement with [4], we have chosen to use Random Fuzzy Variables, proposing to describe the correlation or interaction of repeated measurements by triangular norm based arithmetics.

I. Introduction

The approach followed by the ISO Guide to evaluate measurement uncertainty is basically a statistical approach where the uncertainty is defined as "a parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand" [1]. In [3] the authors have proposed to described extended uncertainty as α-cut and use fuzzy arithmetic based on extension principle of Zadeh to evaluate the uncertainty propagation. We have also seen this is a good method to describe systematic effects, but not random ones. In [4] was proposed a more effective way to estimate measurement uncertainty in terms of Random Fuzzy Variables. But contrary to the classical statistical methods, there is no unique statistical theory on random fuzzy variables. Therefore we propose the t-norm based arithmetics to describe both random and systematic effects on the distribution of measurement results. In statistical approach, the effect of reduction of uncertainty by averaging of series data is a result of assumption that measurements are statistically independent events [5], but when this condition is no more applicable we have to define the right correlation existing. Expression of measurement uncertainty in a random-fuzzy model is the interval with a level of confidence of 1-α (α-cuts), such as (see fig.1):

$$A_\alpha = \{z : \mu_A(z) \geq \alpha\} \tag{1}$$

II. Comparison of t-norms

From a statistical point of view, once measurement results are obtained, we have to choose the estimator of measurand, which is generally the average operator.

So in a probabilistic model we'd average some random variables to get another random variable. It's density function depends on density functions of single input variables (measurement results) and on correlation between measurement acts.

Now for random fuzzy model, if A_1 and A_2 are two measurement fuzzy results (that's to say that their own associated uncertainty are expressed in terms of α-cuts), the indirect measurement algorithm performs the output which is a function "g" of A_1 and A_2, $Z=g(A_1,A_2)$. Evaluation of uncertainty of Z requires a join membership function (taking into account correlations between random contributions to uncertainty) which is defined when a t-norm indicates the principle of joining variables:

$$\mu_{A_1,A_2}(a_1,a_2) = T[A_1(a_1), A_2(a_2)] \tag{2}$$

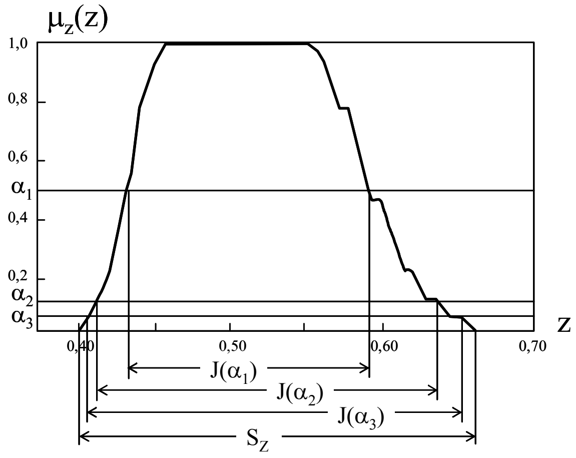

Fig. 1: Example of a membership function with α-cuts marked.

so that the membership function μZ(z) is simply (extension principle):

$$\mu_Z(z) = \sup_{g(a_1,a_2)=z} T[A_1(a_1), A_2(a_2)] \quad (3)$$

Our purpose is to compare several t-norms, as Yager, Dombi, Frank, and Algebraic t-norm.

III. Uncertainty of a measurement result

As in [5] measured data $\{x_i\}^N$ is expressed as a vector of fuzzy intervals. To have our result, N measurement are performed; they furnish a vector of pure real numbers. If we use a DSP-based instrument, each "pure" result is affected from uncertainty: the measurement algorithm it performs represents the indirect measurement result which is a function of the previous ones. So we can define an interval built around the 'one point' estimation, in which there is a given probability for this interval to contain the 'real' value.

In [3] we have observed the general shape of the resulting possibility distributions, once our fuzzy expression has been applied on symmetric probability distributions (Gaussian law, Laplace law, triangular law, uniform law), called the optimal possibility distribution. The authors suggests this shape is not easy to handle and a parameterized one would be more interesting: the pseudo triangular possibility distribution.

The pseudo triangular possibility distribution (tpd) is a piece-wise linear approximation of the optimal possibility distribution, completely determined when some parameters are known. The truncated triangular uncertainty expression has been applied on the four most encountered unimodal and symmetric probability laws, of mean value and standard deviation.

For the uniform and triangular law, the fuzzy tpd is such that $\mu A_i(x_i) = 1$ (in the sense that Ai is built in such way that xi belongs it with the maximum possibility) for xi is chosen equal to the mean value of the interval (the measured pure value). As these are two bounded law, no further approximations are required. Now what can we say about the measurement results when data at our disposal just only consist in a vector of "pure" numbers (generally distributed with gaussian law)? For unbounded laws, an approximation is needed to correctly choose the required parameters (as we can see on Table 2 of [3]).

Every result of measurement xi is transformed into a fuzzy set Ai and so in terms of membership functions and to determine them we can go on as follows:

however we take a x which is out of the real interval $[x_{\min}, x_{\max}]$ (where xmin is the minimum measured data and xmax the maximum one), $\mu A_i(x) = 0$.

Nevertheless, we may also have to determine the systematic effects on the measurement results; as it is intrinsic of the concept of systematic effect, it always appears with the same 'impact' during the execution of all the tests. The ith result is such that $x_i - x_i^* = s + a$ ('s' and 'a' are respectively related to systematic and random effects on the measurement results, while x_i^* represents for us the true value, and therefore it is unknown). At this point if the systematic effects on the measurement results is considered to be contained in [0, δ], and the true mesurand value should be stay between $x_i - \delta$ and x_i with the maximum possibility (remember that δ has got its own sign). Now let's define the ith fuzzy set as follows:

$$Supp(A_i) = [x_{\min} - \delta, x_{\max}]$$

$$Ker(A_i) = [x_i - \delta, x_i]$$

Ai is a trapezoid interval So if we have chosen the averaging operation E_T for the fuzzy interval series $\{A_i\}^N$ as an estimator of measurand, in agreement with the Guide, we have to apply extended addition and multiplication based on t-norm.

IV. Fuzzy Intersections: t-norms

The intersection of two fuzzy sets, A and B, is specified by a binary operation on the unit interval. For each element x of the universal set, this function takes as its arguments the memberships of x in the fuzzy sets A and B, and yields the membership grade of the element in the set constituting the intersection of A and B.

A "fuzzy intersection/t-norm" i is a binary operation that satisfies at least the following axioms for all a, b and d in the range [0,1].

Axiom i1: i(a,1) = a (boundary condition)
Axiom i2: b <= d implies i(a,b) <= i(a,d) (monotonicity)
Axiom i3: i(a,b) = i(b,a) (commutativity)
Axiom i4: i(a,i(b,d)) = i(i(a,b),d) (associativity)

Lets call these four axioms the "axiomatic skeleton for fuzzy intersections/t-norms". It is often desirable to restrict the class of fuzzy intersections (t-norms) by considering additional requirements. Three of the most important are:

Axiom i5: i is a continuous function (continuity)
Axiom i6: i(a,a) < a (subidempotency, weaker than "idempotency", the requirement that i(a,a)=a)
Axiom i7: a1 < a2 and b1 < b2 implies i(a1,b1) < i(a2,b2) (strict monotonicity)

Some Classes of Fuzzy Intersections (t-norms)

The fuzzy literature contains many examples of t-norms, which are a generalization of (classical) set intersection. All of these t-norms are (as far as we know) single valued. To be precise: given a set X, a t-norm is a binary function satisfying certain properties. Hence, given two elements of X, call them x, y, then T (x, y) is also an element of X. Note that this is also true in the context of interval-valued fuzzy sets, fuzzy sets of type 2 and other variants. For example, a t-norm which operates on interval-valued fuzzy sets combines two intervals to produce one interval.

The following are examples of some t-norms that are frequently used as fuzzy intersections (each defined for all a,b in [0,1]) and which satisfy the last three axioms.

Standard intersection:	i(a,b) = min(a,b)	(4)
Algebraic product:	i(a,b) = ab	(5)
Bounded difference (Lukasiewicz's t-Norm):	i(a,b) = max(0,a+b-1)	(6)

Dombi's t-Norm:

$$i(a,b) = \begin{cases} 0 & a=0 \quad or \quad b=0 \\ \dfrac{1}{\left(\left(\dfrac{1}{a}-1\right)^p + \left(\dfrac{1}{b}-1\right)^p\right)^{1/p} + 1} & otherwise \end{cases}$$ for p>1

Frank's t-Norm:

$$i(a,b) = \log_s\left[1 + \frac{(s^a - 1)(s^b - 1)}{s - 1}\right]$$ where s>0, and is different from 1

Hamacher's t-Norm:

$$i(a,b) = \frac{ab}{p + (1-p)(a+b-ab)} \qquad \text{for } p<=2$$

Yager 's t-Norm:

$$i(a,b) = 1 - \min\left\{1, \sqrt[p]{(1-a)^p + (1-b)^p}\right\} \qquad \text{with } p>1$$

In the Dombi case [6], increasing the parameter p will increase the emphasis on the smaller membership value so that, for example, one could emphasize a line of reasoning that considered less likely possibilities.

In [7-8] we see some well known continuous t-norms, as the minimum operator T_M, the algebraic product T_P, and the Lukasiewicz t-norm T_L and some results on the addition of fuzzy intervals are compared. T_M is the strongest (greatest) t-norm, defined as $T_M(x,y)=\min(x,y)$. T_W is the weakest (smallest) t-norm:

$$T_W(x,y) = \begin{cases} \min(x,y) & if \quad \max(x,y)=1 \\ 0 & elsewhere \end{cases}$$

A t-norm is called Archimedean if and only if it's subidempotent , so T_M is clearly not Archimedean. Continuous Archimedean t-norms can be divided in two classes: strict t-norms (strictly increasing norms) and nilpotent t-norms. The algebraic is a strict t-norm, and so Frank's, Hamacher's and Dombi's are. From a Corollary described in [8] we know that for the addition based on the minimum operator, the resulting spreads are the sums of incoming spreads, while for the addition based on the weakest t-norm the resulting spreads are the greatest of the incoming spreads. The following inequality holds:

$$\forall a,b \in [0 \quad 1], T_W(a,b) \le T_L(a,b) = \max(a+b-1,0) \le ab \le \min(a,b)$$

Moreover, for any triangular norm T:

$$T_W(a,b) \le T(a,b) \le T_M(a,b)$$

Now let T_1 and T_2 be two t-norms such that:

$$T_1(a,b) \le T_2(a,b) \qquad \forall a,b \in [0 \quad 1]$$

than for any fuzzy quantity A and B the T-sum of these fuzzy intervals is :

$$A \oplus_{T_1} B \le A \oplus_{T_2} B$$

Note that for the addition based on the minimum operator T_M , the resulting spreads are the sums of the incoming spreads (greatest spreads), while for the addition based on the weakest t-norm the resulting spreads are the greatest of the incoming spreads (smallest spreads).

If we can describe the model about phenomena weighing upon each measurement result obtained by a DSP-based instrument, a meaningful choice in terms of t-norms may be made. Nevertheless, this is a quite complex approach, so to make the right choice a trade-off is necessary. First of all we know that increasing the uncertainty interval we reduce an eventual risk in decision making, but costs are so arising. It' obvious that the exact contrary occurs when uncertainty spreads (see [6]) are too narrow. Another important parameter is simplicity in data processing, because a measurement instrument should be able to yield rapidly measurement uncertainty on a measurement result.

So if we refer to Yager t-norm we note that it is very simple to apply, but perhaps this approach supposes a strong correlation among different measurement results. To avoid this inconvenient we have so proposed the use of Dombi, but the algorithm of computation has got an high complexity in general cases.

We know from theory about fuzzy connectivity that new triangular norms can be constructed starting from the ones already known. An idea may be to combine the easiness in computation of min operator and the capacity of Dombi.

V. Conclusions

In this paper we have proposed the comparison of triangular norms to define the arithmetic on fuzzy intervals, according to describe both systematic and random effects on the distribution of measurement results. This approach has the purpose to obtain measurements results which can be considered compatible with ISO Guide method ones.

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