Maximum Likelihood Frequency Detectors for Orthogonal Multicarrier Systems

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Abstract
In this paper we derive a frequency estimation algorithm for a multicarrier system based on the maximum likelihood principle. We propose a frequency detector structure directly based on this algorithm and a reduced complexity structure having the same performance. However, such a detector shows a very small acquisition range due to the particular form of the likelihood function for a multicarrier system. To overcome this limitation, we propose a modified version of the frequency detector which presents a considerably increased acquisition range together with a low augmentation in complexity.

1 Introduction
Orthogonal multicarrier (MC) digital communication systems have become well known during the last two decades [1]. Over the past five years, different studies [2-3] have shown an increasing interest in these modulation schemes. Thanks to improvements in digital signal processing techniques, MC systems have become practical to implement and permit an efficient transmission scheme for different applications such as modems [4] and digital audio broadcasting [5].

However, this modulation technique is very sensitive to the frequency offset between the emitter and the receiver oscillators. In fact, in the presence of a frequency error, the subcarriers making up the MC signal lose their orthogonality and the system performance becomes degraded.

Unlike single carrier (SC) digital transmission systems, few studies have been carried out concerning frequency synchronization for MC systems [6]. The purpose of this paper is to present, for such systems, new frequency detectors which can be used in an automatic frequency control (AFC) loop [7] to synchronize the receiver oscillator.

Assuming an additive white Gaussian noise (AWGN) channel, these frequency detectors are directly derived from a frequency estimation algorithm based on the maximum likelihood (ML) principle. For SC systems similar algorithms have been studied by F.M. Gardner [8].

The paper is organized as follows: the next section briefly describes the general principle of the orthogonal MC system. In Section 3 we derive a ML frequency estimation algorithm leading to a practical frequency detector (FD) implementation. Unfortunately, this FD presents a small frequency acquisition range compared to the total MC signal bandwidth. Section 4 describes a modified version of the proposed FD allowing a considerable increase in the frequency acquisition range together with a small augmentation in complexity. Finally, Section 5 is for our conclusions.

2 System Description
The transmission system considered in this paper is mainly based on the orthogonal frequency division multiplexing (OFDM) technique [9]. The principle consists of having a large number of modulated carriers (each carrying a low bit rate) which are summed for transmission (frequency multiplexing). For maximum spectral efficiency, an overlapping in the spectra of the transmitted carriers is tolerated: in this case an orthogonality condition on the subcarrier frequencies guarantees the absence of crosstalk between modulated subcarriers at the sampling time.

The OFDM transmitted baseband complex signal can be written as

\[ s_k(t) = \sum_{n=-\infty}^{+\infty} \sum_{k=L_1}^{L_2} d_{n,k} g(t-nT)e^{j2\pi f_k t} \]  \hspace{1cm} (1)

with \( L_1 = \frac{N-N_s}{2} + 1 \), \( L_2 = \frac{N+N_s}{2} \) and \( f_k = \frac{k-1}{T} \), where \( T \) is the symbol duration, \( d_{n,k} \) the coded symbol modulating the \( k^{th} \) carrier \( f_k \) at the \( n^{th} \) time slot of period \( T \), \( N \) the total number of carriers, \( N_s \) the number of modulated carriers and \( g(t) \) the rectangular pulse given by

\[ g(t) = \begin{cases} 1 & , t \in [0,T] \\ 0 & , t \text{ elsewhere.} \end{cases} \]  \hspace{1cm} (2)

\( N-N_s \) carriers are not modulated in order to avoid spectrum overlapping at the receiver which makes use of a discrete Fourier transform (DFT) process for demodulation (usually \( N \) is a power of 2 allowing the use of a fast Fourier transform (FFT) algorithm). Figure 1 shows the block diagram of an orthogonal MC digital transmission system: the transmitted data symbols are split by a serial-to-parallel (S/P) conversion block into \( N \) lower bit rate data streams \( d_{n,k} \) each modulating a subcarrier. These subcarriers are

\[ 1 \rightarrow N-N_s \text{ carriers are not modulated so } d_{n,k} = 0, k \in [1,L_1(U)L_2,N]. \]
then converted via an inverse FFT processor into \( N \) time signals. After a parallel-to-serial (P/S) conversion and a digital-to-analog conversion (TX Filter) of the time signal, the complex baseband transmitted signal \( \tilde{s}(t) \) is obtained. The baseband signal modulates a local carrier and originates the transmitted signal \( s(t) \). Assuming an AWGN channel, \( n(t) \) represents the additive noise and \( r(t) \) the received signal. In the receiver the reverse signal processing is performed: a local oscillator demodulates the received signal to generate \( r_0(t) \). An analog-to-digital and a S/P conversion follow, giving \( N \) baseband time signals which are applied to an FFT processor. The FFT block output signals \( p_{n,k}'s \) are P/S converted and the received data symbols are obtained. In Figure 1, the frequency estimation loop which drives the receiver local oscillator is also shown.

Figure 1: Block diagram of an orthogonal MC transmission system.

3 Maximum Likelihood Frequency Estimation

In this section we derive an algorithm for the carrier frequency recovery of an orthogonal MC signal, which is based on a ML estimation.

After introducing the likelihood function for an orthogonal MC signal, we derive a frequency estimation algorithm and a corresponding frequency detector structure. The performance of this detector is given in terms of its characteristic curve. At the end of this section we also propose a practical implementation of this frequency detector having a considerably reduced complexity.

3.1 Likelihood Function

Let us assume an orthogonal MC signal transmitted over an AWGN channel. The received complex signal is

\[
    r(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} d_{n,k} e^{j2\pi f_0 t} g(t - nT)e^{j(2\pi f_0 t + \phi_0)} + n(t)
\]

where \( n(t) \) is the additive noise. The data symbols \( d_{n,k} \), the carrier frequency \( f_0 \) and the carrier phase \( \phi_0 \) are assumed unknown to the receiver.

For a given set of parameters \( \{d_{n,k}, \tilde{f}_0, \tilde{\phi}_0\} \) the log likelihood function (LLF) is

\[
    \Lambda = \frac{2}{N_0} \int_{T_0} R_e(r(t) \tilde{s}^*(t)) dt
\]

where \( N_0 \) is the two-sided noise spectral density, \( T_0 \) is the observation interval and \( \tilde{s}(t) \) is the signal given by

\[
    \tilde{s}(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} d_{n,k} e^{j2\pi f_0 t} g(t - nT)e^{j(2\pi f_0 t + \phi_0)}.
\]

Considering an observation interval \( T_0 = DT \), with \( D \) a positive integer, and substituting (5) into (4) we can write

\[
    \Lambda = \frac{2}{N_0} \sum_{n=0}^{D-1} \sum_{k=-L_1}^{L_1} \left[ R_e(d_{n,k}) R_e(q_{n,k}) - I_m(d_{n,k}) I_m(q_{n,k}) \right]
\]

where \( q_{n,k} \) is given by

\[
    q_{n,k} = \int_{-T_0}^{T_0} r(t)e^{-j2\pi f_0 t} e^{-j(2\pi f_0 t + \phi_0)} dt.
\]

The data dependence of (6) is removed by averaging the LLF over all the possible values of \( d_{n,k} \). Let us consider \( M^2 \)-state QAM complex symbols whose real and imaginary parts are independent and equally likely: by making some mathematical approximations that are valid at low signal-to-noise ratio (SNR) values, we obtain

\[
    \Lambda = \frac{2(M^2 - 1)}{3N_0^2} \sum_{n=0}^{D-1} \sum_{k=-L_1}^{L_1} \left[ (R_e(q_{n,k}))^2 + (I_m(q_{n,k}))^2 \right].
\]

Substituting for (7) and taking \( D = 1 \), the average value (over the noise and \( d_{n,k} \)) of \( \Lambda \) is given by

\[
    E[\Lambda] = \frac{(M^2 - 1)T^2 \sigma_f^2}{3N_0^2 \pi^2} \sum_{m=L_1}^{L_2} \sum_{k=L_1}^{L_2} \frac{1 - \cos(2\pi \Delta f T)}{(\Delta f T + m - k)^2}
\]

where \( \Delta f \) is the frequency difference between \( f_0 \) and \( \tilde{f}_0 \) and \( \sigma_f^2 \) the variance of the data symbols. Figure 2 shows \( E[\Lambda] \) in the case of \( N = 512 \) and \( N_0 = 400 \) plotted as a function of \( \Delta f T \).

We can notice that the LLF assumes an absolute maximum value for \( \Delta f T = 0 \) and possesses other local maximum points at \( \Delta f T = i \), where \( i \) is an arbitrary non zero integer. The optimum \( f_0 \) value is therefore the one which maximizes the LLF.

3.2 Maximum Likelihood Frequency Detector

Because of the local convexity (with respect to the frequency error) of the LLF, we can apply the gradient method to
maximise it. Taking the derivative of (8) with respect to \( f_0 \) we obtain
\[
\frac{\partial \Lambda}{\partial f_0} = \frac{4(M^2 - 1)}{3N^2} \sum_{n=0}^{D-1} \sum_{k=L_1}^{L_2} \text{Re}\{q_{n,k}q^*_{n,k}\} \quad (10)
\]
where \( q_{n,k} \) is given by
\[
q_{n,k} = -j2\pi \int_{nT}^{(n+1)T} t e^{-j2\pi f_0 t} e^{-j(2\pi f_0 t + 2\phi)} dt. \quad (11)
\]

**Figure 2:** Average value of the log likelihood function for \( N = 512 \) and \( N_u = 400 \).

From equation (10), by taking \( D = 1 \) we can derive an iterative algorithm for the optimization of \( f_0 \):
\[
\tilde{f}_{0,n+1} = \tilde{f}_{0,n} + K_n \epsilon_n \quad (12)
\]
where \( K_n \) is a positive constant, \( \tilde{f}_{0,n} \) and \( \tilde{f}_{0,n+1} \) the estimation of \( f_0 \) at the \( n \)th and \( (n+1) \)th iteration respectively, and \( \epsilon_n \) the control signal given by
\[
\epsilon_n = \sum_{k=L_1}^{L_2} \text{Re}\{q_{n,k}q^*_{n,k}\}. \quad (13)
\]

The recursive frequency estimation algorithm given by (12) can be easily implemented by using an automatic frequency control (AFC) loop [7] having a frequency detector delivering the control signal \( \epsilon_n \) given by (13).

A block diagram of a MC demodulator containing an AFC loop based on the maximum likelihood frequency estimation is shown in Figure 3. The complex signal \( q_{n,k} \) is obtained by rotating the received baseband signal \( r_0(t) \) by \(-2\pi f_k t\) and integrating it over a symbol duration \( T \). Similarly, the complex signal \( \tilde{q}_{n,k} \) is obtained by performing the same processing on the signal \( r_0(t) \) multiplied by \(-j2\pi(t - nT)\). Strictly speaking the multiplication of \( r_0(t) \) by \(-j2\pi(t - nT)\) does not lead to the same \( q_{n,k} \) of expression (11). However, it can be proved that the mean value of the so obtained control signal \( E[\epsilon_n] \) is the same as the one resulting from the multiplication of \( r_0(t) \) by \(-j2\pi t\). The correlator block computes the summation over \( k \) of \( \text{Re}\{q_{n,k}q^*_{n,k}\} \) and gives the maximum likelihood frequency detector (MLFD) output \( \epsilon_n \). After low-pass filtering (LPF), this signal is used to control a local oscillator (VCO) and synchronise it with the frequency of the received signal \( r(t) \). Of course, this detector can not be used for phase synchronisation and therefore an additional phase detector and loop filter are required if a coherent demodulation is needed [7].

**Figure 3:** AFC loop with the MLFD.

The characteristic curve of a frequency detector is the average value of its open-loop output signal plotted as a function of the frequency error \( \Delta f \). By substituting (7) and (11) in (13) it is possible to calculate the MLFD characteristic curve which is given by
\[
E[\epsilon_n] = -\frac{\pi^2 \sigma^2}{2} \sum_{m=1}^{L_1} \sum_{k=L_1}^{L_2} \frac{\sin(2\pi \Delta f T)}{(\Delta f T + m - k)^2 + \left(\frac{\pi}{\Delta f T} + 1\right)}.
\]

Figure 4 shows the characteristic curve of the MLFD in the case \( N = 512 \) and \( N_u = 400 \). We notice that the maximum frequency acquisition range is approximately \( \Delta f T = 0.5 \). This is due to the presence of a minimum point at the same \( \Delta f T \) value in the LLF (see Figure 2). Such an acquisition range is not very large compared to the total spectral occupancy of the orthogonal MC signal. In Section 4 we propose a technique to considerably increase this range.
Each integral in expression (7) and (11) is approximated by a sum of discrete time values. Therefore, the banks of integrators of Figure 3 are substituted by two FFT blocks: the first one is the FFT device which delivers the $p_{n,k}$ symbols and the second one is an additional FFT block which operates on the received samples multiplied by a factor $\alpha_k$ (which depends on the $k^{th}$ subcarrier) and delivers the $p_{n,k}$ symbols. In particular we have $\alpha_k = 0$ for the unmodulated subcarriers and $\alpha_k = -j2\pi(k-1)$ for the modulated ones. The correlator operates on the $p_{n,k}$ and $p_{n,k}$ symbols and delivers a control signal given by

$$\epsilon_n = \sum_{k=1}^{L_1} \Re\{p_{n,k}^*p_{n,k}\}. \quad (15)$$

A low-pass filter and a VCO complete the AFC loop as in the previous case. It is important to observe that the practical implementation of the MLFD based on the FFT blocks works like the theoretical version for the commonly chosen values of $N$ and $N_u$ (i.e. $N = 512$ and $N_u = 400$). In fact, in this case, the sampling frequency is sufficiently high ($N/T$) to avoid spectrum overlapping for frequency errors falling inside the frequency detector acquisition range.

### 4 Modified Maximum Likelihood Frequency Detector

The main drawback of the MLFD is its small acquisition range due to the presence of local maximum and minimum points in the LLF (see Figure 2). In fact, these maxima and minima create zero crossings in the derivate of the LLF giving false locking points for the MLFD.

To overcome this limitation, we modify the shape of the LLF by an appropriate processing on the $q_{n,k}$ and $q_{n,k}$ symbols. Let us consider the new symbols $q'_{n,k}$ and $q''_{n,k}$ given by

$$q'_{n,k} = \sum_{i=-L}^{L} h_i q_{n,k-i} \quad (16)$$

and

$$q''_{n,k} = \sum_{i=-L}^{L} h_i q_{n,k-i} \quad (17)$$

where the $h_i$'s are the tap values of a discrete filter of length $2L + 1$. It can be shown\(^2\) that this is equivalent to multiplying the received signal $r(t)$ by a periodic windowing function of period $T$ given by

$$w(t) = \sum_{i=-L}^{L} h_i e^{j2\pi\tau i/T} \quad (18)$$

and to apply the same processing as in Section 3. The modified LLF $\Lambda'$ is expressed as

$$\Lambda' = \frac{2(M^2 - 1)}{3N_0^2} \sum_{n=0}^{D-1} \sum_{k=L_1}^{L_2} \left[ (\Re(q'_{n,k}))^2 + (\Im(q''_{n,k}))^2 \right]. \quad (19)$$

\(^2\)In expression (7) and (11) $f_k$ is assumed to be equal to $\frac{4\pi}{N}$ for $k \in [L_1 - L, L_2 + L]$. 

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**Figure 4:** Characteristic curve of the MLFD for $N = 512$ and $N_u = 400$.

**Figure 5:** AFC loop with the practical implementation of the MLFD.
Substituting for (16) and taking \( D = 1 \) the average value of \( A' \) is given by
\[
E[|A'|] = \frac{(M^2-1)T^2\sigma_a^2}{3N_0^2\pi^2} \sum_{i=-L}^{L} \sum_{n=-L}^{L} h_i h_n^* \sum_{m=-L}^{L} \sum_{k=-L}^{L} f_1(\Delta fT)
\]
where
\[
f_1(\Delta fT) = \frac{1 - \cos(2\pi\Delta fT)}{(\Delta fT + m - k + i)(\Delta fT + m - k + n)}.
\]
By following the same method as in Section 3, we obtain a control signal \( c_n' \) given by
\[
c_n' = \sum_{k=L_1}^{L_2} Re\{c_n'c_{n,k}^*\}
\]
and a modified MLFD characteristic curve \( E[c_n'] \) equal to
\[
E[c_n'] = \frac{T^2\sigma_a^2}{2\pi} \sum_{i=-L}^{L} \sum_{n=-L}^{L} h_i h_n^* \sum_{m=-L_1}^{L_2} \sum_{k=-L_1}^{L_2} f_2(\Delta fT) + f_3(\Delta fT)
\]
where
\[
f_2(\Delta fT) = \frac{\sin(2\pi\Delta fT)}{(\Delta fT + m - k + i)(\Delta fT + m - k + n)}
\]
and
\[
f_3(\Delta fT) = \frac{[\cos(2\pi\Delta fT) - 1][2(\Delta fT + m - k) + n + i]}{2\pi(\Delta fT + m - k + i)^2(\Delta fT + m - k + n)^2}.
\]
In the sequel we assume a filter with three real coefficients \((L = 1)\) with \( h_0 = 1 \) and \( h_1 = h_{-1} \) (to have an odd FD characteristic curve). In this case the equivalent windowing function is
\[
w(t) = 1 + 2h_1 \cos(2\pi t/T).
\]
Figure 6 shows the influence of the choice of \( h_1 \) on the form of the modified MLFD characteristic curve for \( N = 512 \) and \( N_u = 400 \). We observe that a good choice of \( h_1 \) corresponds to the value of 0.5 as there are no zero crossing points or residual oscillations in the characteristic curve. The complete acquisition range of the modified MLFD is given in Figure 7 for \( h_1 = -0.5 \), \( N = 512 \) and \( N_u = 400 \). From the analysis of this figure we can observe that the acquisition range \( |\Delta fT|_{\text{max}} \) is considerably increased and is equal to 401. In the general case we have
\[
|\Delta fT|_{\text{max}} = N_u + 1.
\]
This fact is due to the absence of local maxima and minima in the LLF (20) plotted in Figure 8. A practical implementation of the modified MLFD is shown in Figure 9. The principle is the same as in Figure 5 except for the presence of two signal processors whose outputs\(^3\) are given by
\[
\hat{p}'_{n,k} = \sum_{i=-L}^{L} h_i \hat{p}_{n,k-i}\]
and
\[
\hat{p}'_{n,k} = \sum_{i=-L}^{L} h_i \hat{p}_{n,k-i}.
\]
Figure 10 shows the characteristic curve of the so obtained modified MLFD for \( h_1 = -0.5 \), \( N = 512 \) and \( N_u = 400 \). We notice a reduction of the frequency acquisition range \( |\Delta fT|_{\text{max}} = 113 \) with respect to the case of Figure 7; this is due to the spectrum folding resulting from the sampling of
\(^3\)Because of the sampling at \( N/T \) we have \( \hat{p}_i = p_{(i-1)\mod N} + 1 \) and \( \hat{p}_i = \hat{p}_{(i-1)\mod N} + 1 \).
the received baseband signal at $N/T$. If the received signal is sampled at $f_s$, the acquisition range can be expressed as

$$|\Delta f T|_{max} = \min(N_u + 1, \lambda N - N_u + 1)$$

where $\lambda$ is the oversampling factor equal to $\frac{\Delta f T}{N}$. For the considered case of $N = 512$ and $N_u = 400$, the maximum theoretical frequency acquisition range of $|\Delta f T|_{max} = 401$ can be achieved with an oversampling factor $\lambda = 2$.

The average value of the modified log likelihood function for $h_1 = -0.5$, $N = 512$ and $N_u = 400$ is shown in Figure 8.

The AFC loop with the practical implementation of the modified MLFD is depicted in Figure 9.

5 Conclusions
In this paper we have derived a frequency estimation algorithm for an orthogonal multicarrier system based on the maximum likelihood principle. A frequency detector and a practical low complexity version have been presented showing, however, quite a small acquisition range. A modified version of these detectors has finally been proposed: it shows a considerably increased frequency acquisition range together with a low augmentation in complexity.

The characteristic curve of the practical implementation of the modified MLFD for $h_1 = -0.5$, $N = 512$ and $N_u = 400$ is shown in Figure 10.


References