On the Frequency Scaling in Continuous-Time Modeling

Rik Pintelon, *Fellow, IEEE,* and Istvan Kollár, *Fellow, IEEE*

***Abstract*—When identifying continuous-time systems in the Laplace domain, it is indispensable to scale the frequency axis to guarantee the numerical stability of the normal equations. Without scaling, identification in the Laplace domain is often impossible even for modest model orders of the transfer function. Although the optimal scaling depends on the system, the model, and the excitation signal, the arithmetic mean of the maximum and minimum angular frequencies in the frequency band of interest is commonly used as a good compromise as shown in the following references: J. Schoukens and R. Pintelon, *Identification of Linear Systems: A Practical Guideline to Accurate Modeling* (London, U.K.: Pergamon), R. Pintelon and J. Schoukens, *System Identification: A Frequency Domain Approach*, (Piscataway, NJ: IEEE Press, 2001), and I. Kollár, R. Pintelon, Y. Rolain, J. Schoukens, G. Simon, "Frequency domain system identification toolbox for Matlab: Automatic processes—From data to model," in *Proc. 13th IFAC Symp. System Identification*, Rotterdam, The Netherlands, Aug. 27–29, 2003, pp. 1502– 1506. In this paper, we show: 1) that the optimal frequency scaling also strongly depends on the estimation algorithm and 2) that the median of the angular frequencies is a better compromise for improving the numerical stability than the arithmetic mean.**

***Index Terms*—Continuous-time modeling, frequency domain, system identification.**

I. Problem Statement

CONSIDER the identification of rational transfer function models in the Laplace variable s

$$G(s,\theta)=\frac{B(s,\theta)}{A(s,\theta)}=\frac{\sum_{n=0}^{n_b} b_n s^n}{\sum_{n=0}^{n_a} a_m s^m}$$

with

$$\theta=[a_0,a_1,\ldots,a_{n_a},b_0,b_1,\ldots,b_{n_b}]^T \qquad (1)$$

starting from measured input/output spectra $U(k),Y(k)$ or frequency response functions $G(j\omega_k), k=1,2,\ldots,F$. The goal is to estimate the numerator $b_0,b_1,\ldots$ and denominator $a_0,a_1,\ldots$ coefficients for a given value of the order of the numerator n_b and denominator n_a polynomials. Since parametrization (1) is not identifiable ($G(s,\lambda\theta)=G(s,\theta)$ for any nonzero real number λ), the parameter vector θ should be constrained, for example, $a_0=1$ or $b_{n_b}=1$. From a numerical point of view, it is often better to use the full overparametrized form (1) in combination with the 2-norm constraint $\theta^T\theta=1$ (see [2, ch. 18]). This approach will be used throughout the paper.

Most algorithms estimate θ by minimizing (in each step) a "quadratic-like" cost function $V(\theta,Z)$

$$V(\theta,Z)=\varepsilon^H(\theta,Z)\varepsilon(\theta,Z)=\sum_{k=1}^{F}|\varepsilon(s_k,\theta,Z(k))|^2 \qquad (2)$$

where superscript H is the Hermitian transpose (transpose + complex conjugate), and $s_k=j\omega_k$. The F by 1 vector of the residuals $\varepsilon(\theta,Z)$ is some kind of measure of the difference between the measurements and the model. $\varepsilon_{[k]}(\theta,Z)=\varepsilon(s_k,\theta,Z(k))$ is a (non)linear function of the model parameters θ and the measurements $Z(k)$ at frequency k (input/output spectra $U(k),Y(k)$ or frequency response functions $G(j\omega_k)$).

Often, a Newton-Gauss type algorithm [4] is used to find the minimizer $\hat{\theta}(Z)$ of (2). Rewriting (2) as $V(\theta,Z)=\varepsilon_{\rm re}^T(\theta,Z)\varepsilon_{\rm re}(\theta,Z)$, where $(\,)_{\rm re}$ stacks the real and imaginary parts on top of each other,

$$\varepsilon_{\rm re}(\theta,Z)=\begin{bmatrix}\mathrm{Re}(\varepsilon(\theta,Z))\\ \mathrm{Im}(\varepsilon(\theta,Z))\end{bmatrix} \qquad (3)$$

the ith iteration step of this algorithm is given by

$$J_{\rm re}^T\left(\theta^{(i-1)},Z\right)J_{\rm re}\left(\theta^{(i-1)},Z\right)\Delta\theta^{(i)}=-J_{\rm re}^T\left(\theta^{(i-1)},Z\right)\varepsilon_{\rm re}\left(\theta^{(i-1)},Z\right) \qquad (4)$$

with $\Delta\theta^{(i)}=\theta^{(i)}-\theta^{(i-1)}$ and $J(\theta,Z)=\partial\varepsilon(\theta,Z)/\partial\theta$ the Jacobian of the vector $\varepsilon(\theta,Z)$. The numerical stability (sensitivity) of the solution of the normal equation (4) is basically limited by the numerical precision of the calculation of $J_{\rm re}^TJ_{\rm re}$. The latter is quantified by the condition number κ of $J_{\rm re}^TJ_{\rm re}$, where $\kappa(J_{\rm re}^TJ_{\rm re})=\kappa^2(J_{\rm re})$ [5]. The solution of (4) can be calculated without forming the product $J_{\rm re}^TJ_{\rm re}$ explicitly by solving the overdetermined set of equations

$$J_{\rm re}\left(\theta^{(i-1)},Z\right)\Delta\theta^{(i)}=-\varepsilon_{\rm re}\left(\theta^{(i-1)},Z\right) \qquad (5)$$

using a singular value decomposition or a QR-factorization of the matrix $J_{\rm re}$ (see [2], [5]). The numerical precision of (5) is basically limited by $\kappa(J_{\rm re})$. Summarized, the solution of (4) and (5) is numerically reliable if, respectively,

$$\log_{10}(\kappa^2(J_{\rm re}))\ll d \quad\text{and}\quad \log_{10}(\kappa(J_{\rm re}))\ll d \qquad (6)$$

Manuscript received July 11, 2003; revised March 16, 2004. This work was supported by the Fund for Scientific Research (FWO-Vlaanderen), by the Flemish Government (GOA-IMMI), and by the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy programming (IUAP V/22).

R. Pintelon is with the Electrical Measurement Department, Vrije Universiteit Brussel, 1050 Brussels, Belgium (e-mail: Rik.Pintelon@vub.ac.be).

I. Kollár is with the Management Information Systems Department, Budapest University of Technology and Economics, 1521 Budapest, Hungary.

Digital Object Identifier 10.1109/TIM.2004.838916

with d the number of significant digits used in the calculations. Hence, in general, solving (5) should be preferred over solving (4).

When identifying continuous-time models (1), it is indispensable to scale the frequency axis (and, hence, also the parameters) to guarantee condition (6). Without scaling, identification in the s-domain is often impossible with the available computing precision 10^{-d}, even for modest orders of the transfer function (see [1] and Section II-B). Although the scale factor which minimizes $\kappa(J_{\rm re})$ is plant, model, and excitation dependent, a good compromise is to use the arithmetic mean of the maximum and minimum angular frequencies in the frequency band of interest [1]–[3]

$$\omega_{\rm scale} = \frac{\omega_{\max} + \omega_{\min}}{2} \equiv \omega_{\rm ari} \text{ with } \begin{cases} \omega_{\min} = \min\{\omega_1, \omega_2, \ldots \omega_F\} \\ \omega_{\max} = \max\{\omega_1, \omega_2, \ldots \omega_F\}. \end{cases} \quad (7)$$

For example, the term $a_m s^m$ becomes after scaling $a_m \omega_{\rm scale}^m (s/\omega_{\rm scale})^m$ and $a_m \omega_{\rm scale}^m$ is estimated. Expanding the numerator $B(s,\theta)$ and denominator $A(s,\theta)$ polynomials of transfer function model (1) in two scalar orthogonal polynomials bases (see [2], [6]) or in one vector orthogonal polynomial basis (see [2], [7]), improves the numerical conditioning further at the cost of an increased calculation time. However, the problem that the numerical stability of the procedure (strongly) depends on the choice of $\omega_{\rm scale}$ remains.

The contribution of this paper is to show that: 1) the optimal scaling that minimizes the condition number of the normal equation (5) strongly depends on the estimation algorithm used, and 2) the median of the angular frequencies

$$\omega_{\rm scale} = \text{median}\{\omega_1, \omega_2, \ldots, \omega_F\} \equiv \omega_{\rm med} \quad (8)$$

is a better compromise for reducing the condition number than the arithmetic mean (7).

Notes

1) $\omega_{\rm med} = \omega_{\rm ari}$ for uniformly distributed frequencies, while they are different for nonuniformly distributed frequencies. Two practical examples of nonuniformly distributed frequencies are multiple experiments with overlapping frequency bands and logarithmic frequency distributions. The latter are used when measuring a device over several frequency decades, for example, electrical machines, operational amplifiers, electrochemical processes, traction batteries, and loudspeakers [1], [2].
2) The scaling factors (7) and (8) make sense only if a "reasonable" frequency set has been chosen for the experiment. It should excite the relevant frequency regions [for example, the passband(s)] of the system to be identified. We do not consider the cases where this good engineering practise has been violated.
3) The results of the paper remain valid for rational forms in $\sqrt{s}$ which are well suited for modeling diffusion phenomena [2], [8]. For such models, $a_m(\sqrt{s})^m$ becomes after scaling $a_m w_{\rm scale}^m (\sqrt{s}/w_{\rm scale})^m$ where $w_{\rm scale}$ is calculated as in (7) and (8), but now on the set $\{w_1, w_2, \ldots, w_F\}$ with $w = \sqrt{\omega}$.
4) An identification procedure is numerically stable if the condition number of the starting values and the condition number of each iteration step of the nonlinear minimization (5) satisfies (6).
5) Numerical conditioning is usually not a problem in discrete-time modeling [replace s by z^{-1} in (1)] provided that the frequencies are uniformly distributed over the unit circle ($z = e^{j\omega T_{\rm s}}$ with $T_{\rm s}$ the sampling period). The reason for this is that the powers of z are orthogonal functions over the unit circle.

II. Median Frequency Scaling

First, an intuitive explanation for the median scaling $\omega_{\rm med}$ is given (Section II-A), and next, its performance is compared with the arithmetic scaling $\omega_{\rm ari}$ on real measurement data for several estimation algorithms (Section II-B). The estimation algorithms considered are the maximum-likelihood estimator (ML) and starting value generators for the ML algorithm: the iterative quadratic maximum likelihood (IQML), the total least squares (TLS), the generalized total least squares (GTLS), and the bootstrapped total least squares (BTLS). All these estimators minimize a cost function of the form (2), and the reader is referred to [2] for a detailed description of the algorithms. Since the TLS estimates are biased and overemphasize high frequency errors (see [2]), the GTLS, IQML, and BTLS estimates are usually preferred for generating the starting values. The condition numbers of one step of the IQML and BTLS algorithms started from the GTLS solution are shown. For the ML estimator, the condition number of the iterative algorithm (5) at convergence is given. Note, however, that the condition number of (5) varies over the iteration steps.

A. Rationale

Assume that the angular frequencies are distributed according to some density $n(\omega)$. The total number of frequencies $N(\omega)$ in the band $[\omega_{\min}, \omega]$ equals then

$$N(\omega) = \int_{\omega_{\min}}^{\omega} n(\omega)\, d\omega. \quad (9)$$

Since the "number-of-frequencies domain" $N(\omega)$ is equispaced, the arithmetic rule is a good choice for normalizing the number of frequencies $N(\omega)$, giving $F/2$, where $F = N(\omega_{\max})$ is the total number of frequencies in the band $[\omega_{\min}, \omega_{\max}]$. The scaling (8) is now found by solving $N(\omega_{\rm scale}) = F/2$

$$N(\omega_{\rm scale}) = F/2 \Rightarrow \omega_{\rm scale} = N^{-1}(F/2) \quad (10)$$

where $N^{-1}(F/2)$ is precisely the median of the frequency distribution [9]. For discrete frequency distributions ($n(\omega)$ is a sum of Dirac impulses) $N^{-1}(F/2)$ simplifies to

$$\omega_{\rm scale} = \text{median}\{\omega_1, \omega_2, \ldots, \omega_F\}. \quad (11)$$

Note that the median operator detects where the bulk of the frequencies lies, even in the case of experiments with repeated frequencies and/or overlapping frequency bands.

B. Examples

The performance of $\omega_{\rm med}$ and $\omega_{\rm ari}$ is illustrated on real measurement data with a uniform (flexible robot arm, see Fig. 1) and

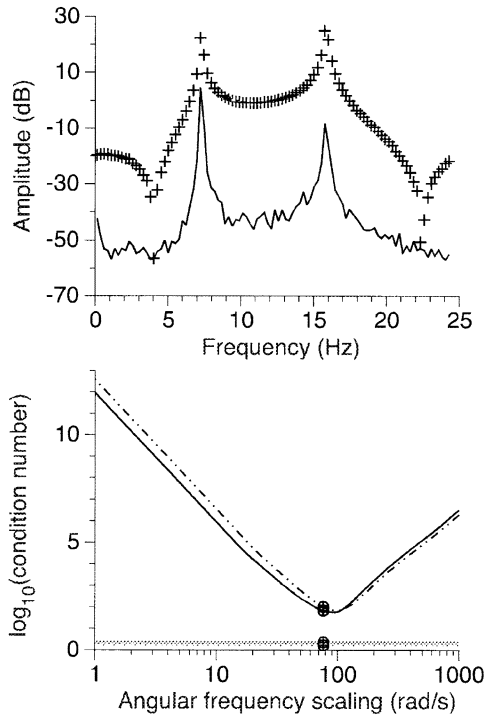


Fig. 1. Flexible robot arm. Top: measured driving couple to acceleration frequency response function ("+") and its standard deviation (solid line). Bottom: condition number identified model [(1) with $n_b = 4, n_a = 6$] as a function of the scaling $\omega_{\rm scale}$ for different estimators. Black solid line: ML and IQML estimate; dash dot-dot line: TLS estimate; gray solid line: GTLS estimate; gray dotted line: BTLS estimate; and "+", "o" the condition numbers corresponding to $\omega_{\rm med}$ and $\omega_{\rm ari}$, respectively.

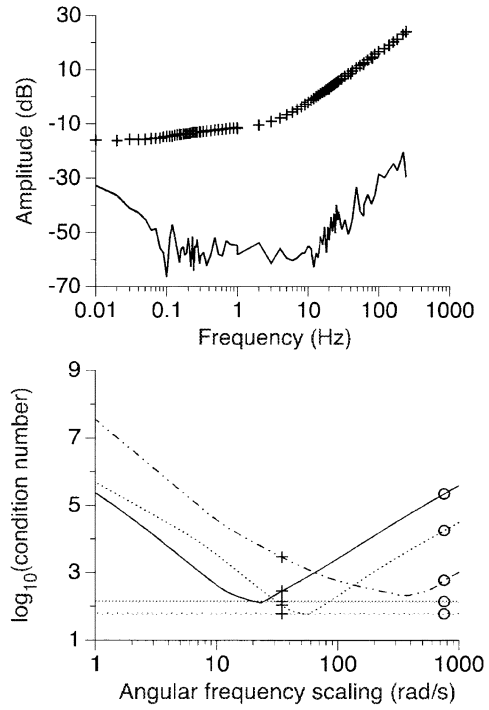


Fig. 2. Electrical machine. Top: measured impedance ("+") and its standard deviation (solid line). Bottom: condition number identified model [(1) with $n_b = 3, n_a = 2$] as a function of the scaling $\omega_{\rm scale}$ for different estimators. Black solid line: ML estimate; black dotted line: IQML estimate; dash dot-dot line: TLS estimate; gray solid line: GTLS estimate; gray dotted line: BTLS estimate; and "+", "o" the condition numbers corresponding to $\omega_{\rm med}$ and $\omega_{\rm ari}$, respectively.

TABLE I
FLEXIBLE ROBOT ARM: ESTIMATES OF MODEL (1) WITH $n_b = 4, n_a = 6$

	ML	IQML	TLS	GTLS	BTLS
a_0	2.3110e-01	2.3098e-01	2.3122e-01	2.3069e-01	2.3107e-01
a_1 (s/rad)	1.2093e-04	1.2112e-04	1.1890e-04	1.1944e-04	1.2127e-04
a_2 (s/rad)^2	1.3793e-04	1.3789e-04	1.3798e-04	1.3800e-04	1.3791e-04
a_3 (s/rad)^3	5.5722e-08	5.5808e-08	5.4591e-08	5.5133e-08	5.5869e-08
a_4 (s/rad)^4	1.3558e-08	1.3578e-08	1.3505e-08	1.3468e-08	1.3564e-08
a_5 (s/rad)^5	3.7403e-12	3.7486e-12	3.6432e-12	3.6866e-12	3.7523e-12
a_6 (s/rad)^6	2.0061e-13	2.0279e-13	1.9485e-13	1.9038e-13	2.0131e-13
b_0	-2.5738e-02	-2.5671e-02	-2.5941e-02	-2.6250e-02	-2.5718e-02
b_1 (s/rad)	-9.5516e-06	-9.4410e-06	-1.5375e-05	-1.4862e-05	-9.4676e-06
b_2 (s/rad)^2	-4.1654e-05	-4.1558e-05	-4.2054e-05	-4.2458e-05	-4.1626e-05
b_3 (s/rad)^3	-2.2615e-09	-2.2550e-09	-2.6413e-09	-2.6501e-09	2.2653e-09
b_4 (s/rad)^4	-2.0263e-09	-2.0219e-09	-2.0465e-09	-2.0666e-09	-2.0250e-09
ML cost function	247.5	248.7	274.2	352.0	247.6

TABLE II
ELECTRICAL MACHINE: ESTIMATES OF MODEL (1) WITH $n_b = 3, n_a = 2$

	ML	IQML	TLS	GTLS	BTLS
a_0	3.2842e-02	8.1608e-02	9.0089e-01	5.5680e-01	6.5894e-02
a_1 (s/rad)	2.4848e-02	2.5313e-02	2.1294e-03	-2.0740e-02	2.5098e-02
a_2 (s/rad)^2	1.6549e-04	8.7072e-05	9.7601e-07	1.6143e-04	1.2584e-04
b_0	5.3111e-03	1.4378e-02	1.9472e-01	1.1610e-01	1.1285e-02
b_1 (s/rad)	6.6084e-03	7.6113e-03	1.0995e-02	-7.8925e-03	7.2207e-03
b_2 (s/rad)^2	3.4005e-04	3.2048e-04	2.2122e-05	-1.8065e-04	3.3005e-04
b_3 (s/rad)^3	1.7857e-06	9.2110e-07	1.1035e-08	1.6618e-06	1.3494e-06
ML cost function	6.67e+03	1.78e+04	8.70e+04	1.33e+06	7.76e+03

a quasilogarithmic (electrical machine, see Fig. 2) frequency distribution. A detailed description of the measurement set up can be found in [2], the data files are available in [10], and the estimated model parameters are listed in Tables I and II. Figs. 1 and 2 show the condition number of the normal (5) as a function of the scaling $\omega_{\rm scale}$. For ML, IQML, and TLS it is the condition number of the corresponding Jacobian matrix $J_{\rm re}$; while for GTLS and BTLS it is the condition number of the generalized singular value decomposition of the corresponding matrix pair $(J_{\rm re}, C_{J_{\rm re}}^{1/2})$, where $C_{J_{\rm re}}$ is the column covariance matrix of $J_{\rm re}$ (if $C_{J_{\rm re}}$ is regular then it equals the condition number of $J_{\rm re}C_{J_{\rm re}}^{-1/2}$). The following observations can be made from Figs. 1 and 2.

1) Without scaling $(\omega_{\rm scale} = 1)$ the condition number κ of the ML, IQML, and TLS estimates of the flexible robot arm model (see Fig. 1) is that large $(>10^{12})$ that the solution of (5) is unreliable within the numerical precision of calculations ($d = 14$ significant digits).
2) Within reasonable limits the condition numbers of the GTLS and BTLS algorithms are insensitive to the scaling

ω_{scale}. Usually, they are also (well) below that of the ML estimator.

3) The optimal scaling (resulting in the minimal condition number) of the ML, IQML, and TLS estimates are significantly different for the nonuniform frequency distribution (see Fig. 2). The deviation is most apparent for the TLS estimates which can be explained by the fact that it overemphasizes high frequency errors (see [2]).
4) For all estimators except the TLS, the median scaling (8) outperforms the arithmetic scaling (7) on the nonuniform frequency distribution.

These observations are confirmed by several other measurement data sets and by extensive simulations.

III. Conclusion

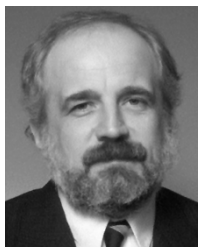
It has been shown that the optimal frequency scaling strongly depends on the estimation algorithm. Except for the TLS estimator, the median scaling outperforms the arithmetic mean for nonuniform frequency distributions (e.g., logarithmic frequency distributions, and multiple experiments with overlapping frequency bands and/or repeated frequencies). As such, the median scaling allows one to identify higher order problems.

References

[1] J. Schoukens and R. Pintelon, *Identification of Linear Systems: A Practical Guideline to Accurate Modeling*, London, U.K.: Pergamon, 1991.
[2] R. Pintelon and J. Schoukens, *System Identification: A Frequency Domain Approach*. Piscataway, NJ: IEEE Press, 2001.
[3] I. Kollár, R. Pintelon, Y. Rolain, J. Schoukens, and G. Simon, "Frequency domain system identification toolbox for Matlab: Automatic processes—From data to model," in *Proc. 13th IFAC Symp. System Identification*, Rotterdam, The Netherlands, Aug. 27–29, 2003, pp. 1502–1506.
[4] R. Fletcher, *Practical Methods of Optimization*, 2nd ed. New York: Wiley, 1991.
[5] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD: The Johns Hopkins Univ. Press, 1996.
[6] Y. Rolain, R. Pintelon, K. Q. Xu, and H. Vold, "Best conditioned parametric identification of transfer function models in the frequency domain," *IEEE Trans. Autom. Control*, vol. 40, no. 11, pp. 1954–1960, Nov. 1995.
[7] R. Pintelon, Y. Rolain, A. Bultheel, and M. Van Barel, "Numerically robust frequency domain identification of multivariable systems," in *Proc. 26th Int. Seminar Modal Analysis*, Leuven, Belgium, Sep. 16–18, 2002, pp. 1315–1321.
[8] R. Pintelon, "Identification of linear time invariant diffusion phenomena," *IEEE Trans. Instrum. Meas.*, vol. 47, no. 5, pp. 1053–1055, Oct. 1998.
[9] A. Stuart and J. K. Ord, *Kendall's Advanced Theory of Statistics—Distribution Theory*, London, U.K.: Griffin, 1987, vol. 1.
[10] Frequency Domain System Identification Toolbox for Use with Matlab, by I. Kollár, G. Simon, and G. Román. [Online]. Available: http://elecwww.vub.ac.be/fdident/

Rik Pintelon (M'90–SM'96–F'98) was born in Gent, Belgium, on December 4, 1959. He received the Electrical Engineer (Burgerlijk ingenieur) degree in July 1982, the Doctor in Applied Sciences Degree in January 1988, and the qualification to teach at university level (Ggeaggregeerde voor het Hoger Onderwijs) in April 1994, all from the Vrije Universiteit Brussel (VUB), Brussels, Belgium.

From October 1982 to September 2000, he was a Researcher at the Fund for Scientific Research, VUB, Flanders, Belgium. Since October 2000, he has been a Professor at the VUB in the Electrical Measurement Department (ELEC). His main research interests are in the fields of parameter estimation/system identification and signal processing.



Istvan Kollár (M'87–SM'93–F'97) was born in Budapest, Hungary, in 1954. He received the M.S. degree in electrical engineering from the Technical University of Budapest, Budapest, Hungary, in 1977, and the "Candidate of Sciences" and "Doctor of the Academy" degrees from the Hungarian Academy of Sciences, Budapest, Hungary, in 1985 and 1998, respectively.

From 1989 to 1990, he was a Visiting Scientist at the Vrije Universiteit Brussel, Brussels, Belgium. From 1993 to 1995, he was a Fulbright Scholar and Visiting Associate Professor in the Department of Electrical Engineering, Stanford University, Stanford, CA. Currently, he is a Professor of electrical engineering at the Budapest University of Technology and Economics, Budapest, Hungary. His research interests span the areas of digital and analog signal processing, measurement theory, and system identification. He has published about 70 scientific papers, and is coauthor of the book T*echnology of Electrical Measurements* (New York: Wiley, 1993). He authored the Frequency Domain System Identification Toolbox for Matlab. He was Editor of *Periodica Polytechnica* Series of Electrical Engineering from 1988 to 1991, then Chief Editor of *Periodica Polytechnica* from 1991 to 1997. He was Technical Cochairman of IMTC/2001.

Dr. Kollár is a member of the Administrative Committee of the IM Society. He is also an active member of EUPAS in IMEKO.