Signal Enhancement of Nonsynchronized Measurements for Frequency Domain System Identification

István Kollár, *Member, IEEE*

Abstract—**The possibilities of an *a posteriori* synchronization of nonsynchronized data records are investigated for the purpose of signal enhancement of a periodic signal. The theoretical background for an effective frequency domain method is given, and the errors of the averaged complex amplitudes are calculated. The time delay between records is identified via maximum likelihood estimation, which is slightly modified in order to improve computational effectiveness. The results are supported by simulations.**

Keywords—**Signal processing, noise suppression, signal enhancement, ensemble averaging, synchronization, time delay estimation, system identification.**

I. Introduction

THE most straightforward way to improve the signal-to-noise ratio (SNR) of measured signals is signal enhancement, that is, the averaging of repetitive measurements. In order not to change the signal waveform by this averaging, the useful signal must be the same and identically positioned in each time record; that is, measurements must be *synchronized* to the signal, by using an appropriate trigger.

However, such a synchronization is not always realizable. An example is the measurement of the transfer function of a physically large object (e.g., a transmission line, see [1]), where the recorder has to measure the system response to a periodic excitation, without a synchronization signal available. Another example is a digital oscilloscope with posttriggering. In this case the measurements can be synchronized only to an uncertainty of about half of the sampling period.

A similar problem arises in practically every A/D interface. The sampling units are usually driven by a high-frequency, but nonsynchronized clock, and the trigger starts periodic sampling at the *next* clock pulse. This small uncertainty (half of the period length of the high-frequency clock) is generally tolerable; however, sometimes it may be desirable to decrease its effect.

The situation is similar again when the excitation signal is repetitive, and sampling is performed with a stable but not synchronized clock. Although in a long time record the known repetition rate can be used as a basis for averaging, a slow slip between the generator and the recorder clock may quickly result in signal deterioration during averaging [1].

In the above situations it is essential for signal enhancement to find an algorithm to synchronize the more or less randomly timed measurements. In the next section we discuss the possibilities for synchronization, with a proposed effective algorithm.

II. Possibilities for Signal Synchronization

The above problem reduces to the *estimation of the time delay* between two identical but unknown waveforms, covered by noise. Time delay estimation has extensive literature [2]–[9]. In telecommunications usually either sinusoidal or nearly sinusoidal signals are to be synchronized (carrier synchronization), or the delay between random signals is sought. Often multipath propagation channels are investigated. In time domain reflectometry the delay between (usually modified) pulses is sought.

In the case of frequency domain system identification [10]–[11] the applied model can be simpler. The system under test is usually linear, and a periodic excitation is often applied. Consequently, the output signal consists of harmonically related sinusoids and additive noise. In this case, results concerning delay estimation can be obtained in rather simple forms for sampled, finite length records.

The common way to find the time delay between two similar but noisy signals is the evaluation of the cross-correlation function, because its maximum marks good coincidence of the two signals. However, the resolution of digital cross-correlation is restricted by the sampling interval. In the case of proper sampling, interpolation is also possible. For high SNR values the cross-correlation function has to be evaluated on a very fine grid.

If it is known that the useful content of the signals is exactly the same, a least squares (LS) approach can also be followed:

$$\hat{\tau}_{LS} = \arg\min \{C_{LS}(\tau)\} = \arg\min_{\tau} \left\{ \sum_{i=0}^{M-1} (x_1(t_i) - x_2(t_i - \tau))^2 \right\}. \quad (1)$$

It is easy to show that the solution is the same as that of

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The author is with the Department of Measurement and Instrument Engineering, Technical University of Budapest, H-1521 Budapest, Hungary.

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the cross-correlation method, if an integer number of periods is measured, and the sampling frequency is higher than the Nyquist rate of both signal and noise.

When discussing the accuracy limitations of the above time delay estimation methods, a paradox exists; that is, signal enhancement is needed for the improvement of the SNR, but *it is just the small SNR* which limits the possibility of accurate synchronization, necessary for useful signal enhancement.

Now the question can be posed as follows. *Is it possible to improve the SNR before enhancement*? Fortunately, the answer is yes for periodic signals (and also for band-limited ones). In the frequency domain the power of periodic signals is concentrated at discrete points, while the noise power is usually distributed in a wide frequency band. Thus, around the frequency domain signal peaks the SNR is much better than in the time domain. Moreover, frequency domain has another important advantage. Time delay appears as a multiplicative term $\exp(-j2\pi f\tau)$, which contains τ in analytical form; thus optimization is much easier.

In the following sections we are going to analyze the possibility of synchronization of the Fourier transforms of the signals. Because of the one-to-one relationship between the time domain signal and its Fourier transform, signal enhancement can be performed in the frequency domain, and the average can be transformed back, or used directly as in frequency domain system identification [10]–[11].

III. Formulation of the Frequency Domain Synchronization Problem

Let the ideal time domain signal be $x(t)$. The two measurements are corrupted by noises $n_1(t)$ and $n_2(t)$. The result of the sampled measurements is two sets of noisy data points:

$$z_{1i} = x(i\Delta t) + n_1(i\Delta t),$$
$$z_{2i} = x(i\Delta t + \tau) + n_2(i\Delta t), \qquad i = 0, 1 \cdots M-1. \tag{2}$$

In what follows, Δt will be chosen as 1, for the sake of simplicity.

Let us assume that the noise samples form zero mean white Gaussian series with standard deviation σ_n. The discrete Fourier transforms are:

$$Z_{1r} = X(r\Delta f) + N_1(r\Delta f),$$
$$Z_{2r} = e^{-j2\pi r\Delta f\tau}X(r\Delta f) + N_2(r\Delta f),$$
$$r = 0, 1 \cdots M-1 \tag{3}$$

where $\Delta f = 1/M$.

In these expressions the noise terms for $r \neq 0$ are independent, zero mean, complex Gaussian random variables; the variances of the independent real and imaginary parts are equal to

$$\sigma^2 = \frac{M}{2}\sigma_n^2. \tag{4}$$

This is a well-defined estimation problem for the parameter τ, given the observed data Z_{1r}, Z_{2r}. A maximum likelihood estimate can be formulated, based on the above expressions. This problem is already solved in the literature [10]–[11]. Let us notice that Z_1 and Z_2 can be considered as noisy observations of the input and output signals of a system whose transfer function is a pure delay. Thus, the frequency domain system identification procedure ELiS [10]–[11] can be used. Though the estimation formulation is still nonlinear in τ, only one parameter is to be estimated; thus the iteration is quick. There is just one problem that has to be tackled. The cost function as a function of τ has several local minima, and we need just the best one. This can be found by first scanning the possible range of τ in reasonably small steps (e.g., 1/2–1/4 part of the period length of the highest significant frequency component), and choosing those values where the cost function is small. The iteration is to be started from each of these values. The delay with the smallest cost function will be chosen.

This procedure is theoretically sound; however, the calculations may take considerable computing time. Thus, it is reasonable to make use of the special form of the transfer function, and, if possible, try to find a quicker procedure.

The general cost function of ELiS, to be minimized via the transfer function parameters, is [10]–[11]

$$K = \sum_{k=1}^{F} \frac{|z_k^{-\tau_d}N(z_k^{-1})X_{mk} - D(z_k^{-1})Y_{mk}|^2}{2\sigma_{xk}^2|N(z_k^{-1})|^2 + 2\sigma_{yk}^2|D(z_k^{-1})|^2}, \tag{5}$$

where

$$z_k = e^{j2\pi f_k\Delta t}, \quad \tau_d = \tau f_s. \tag{6}$$

X_m and Y_m are the measured input and output complex amplitudes (that is, the Fourier transforms of the measured input and output signals at the frequencies of interest), respectively, and the transfer function is sought in the form

$$z^{-\tau_d}\frac{N(z^{-1})}{D(z^{-1})}. \tag{7}$$

Let us notice that summation is performed for selected frequencies only; this is the step where the SNR can be significantly improved.

It is easy to see that in our case, since $N(z^{-1}) \equiv 1$ and $D(z^{-1}) \equiv 1$, the cost function simplifies to

$$K = \sum_{k=1}^{F} \frac{|z_k^{-\tau_d}Z_{1k} - Z_{2k}|^2}{4\sigma^2}. \tag{8}$$

If the noise is much smaller than the signal amplitudes (which is the usual case in the frequency domain), and τ

is close to the true value, a simple approximation can be used:

$$|z_k^{-\tau_d} Z_{1k} - Z_{2k}|^2 \approx |Z_{1k}|^2 |(\arg(Z_{1k}) - 2\pi f_k \tau - \arg(Z_{2k})) \pmod{2\pi}|^2. \quad (9)$$

Thus the minimization of (8) becomes a linear LS problem, which can be solved by effective methods. If τ is not close to its true value, (9) and the terms in (8) differ significantly, but they still have their minima at the same place. The only difficulty is phase wrapping [12] which can be overcome by scanning the possible values of τ on a sufficiently dense grid, followed by the solution of the LS problem, with iterations started from the promising values [11].

For fair comparison with the other methods, it should be mentioned that the SNR can be improved also in the case of the cross-correlation method or the time domain LS method, if it is known that the useful signal consists of a few harmonically related sine waves [9]. Thus, before calculating the cross-correlation, both signals can be transformed via FFT, the frequencies of the sine waves can be selected, the other points of the frequency functions can be set to zero, and after inverse FFT the cross-correlation can be calculated. The maximum value can also be sought in the frequency domain, since

$$\arg\max_{\tau} \{\hat{R}_{x_1x_2}(\tau)\} = \arg\max_{\tau} \left\{ \sum_{k=1}^{F} e^{j2\pi f_k \tau} \overline{X_{1k}} X_{2k} \right\}. \quad (10)$$

IV. Variance of the Estimate

The variance of the estimate can be studied by calculating the Cramér–Rao lower bound. It is well known that for maximum likelihood estimates this bound is given by

$$\text{var}\{\hat{\tau}\} \geq 1 \Big/ \left(\frac{\partial^2 K}{\partial^2 \tau} \right) \Bigg|_{\tau = \tau_{\text{true}}, Z_{1k} = X_k, Z_{2k} = \exp(-j2\pi f_k \tau) X_k}, \quad (11)$$

which gives in our case for (8):

$$\text{var}\{\hat{\tau}\} \geq \frac{2\sigma^2}{\sum_{k=1}^{F} |X_k^2| (2\pi f_k)^2} = \frac{2\sigma_n^2}{\sum_{k=1}^{F} M \frac{A_k^2}{2} (2\pi f_k)^2}, \quad (12)$$

where

$$|X_k| = \frac{M}{2} A_k. \quad (13)$$

A_k is the amplitude of the harmonic component of the time function at frequency f_k. This result is in conformity with [4].

(12) is valid for *multisines* (periodic signals with a limited number of harmonic components). Thus, the performance of frequency domain synchronization can be investigated for this often occurring case [10]–[11]. The larger the number of sinusoidal components, and the higher the frequency of the components versus the sampling frequency, the smaller the variance will be.

The lower bound is approximately reached if (9) is a good approximation, which is true if the SNR is sufficiently high [7].

V. The Effect of Imperfect Synchronization on the Averaged Complex Amplitudes

Imperfect synchronization results in an additional uncertainty of the averaged complex amplitudes. The effect on the estimated phase and absolute value can be easily calculated.

The *phase* is in linear relationship with the delay: $\varphi_k = 2\pi f_k \tau$. Thus the mean value is not distorted, and the standard deviations are related similarly:

$$\text{std}\{\hat{\varphi}_k\} = 2\pi f_k \,\text{std}\{\hat{\tau}\}. \quad (14)$$

The effect on the *amplitude* estimates is more complicated. While an individual delay value does not change the amplitude, the average of complex amplitudes with different phases will be slightly distorted. The bias can be calculated as follows:

$$b\{\hat{X}_k\} = X_k \left(1 - \int_{-\infty}^{\infty} \cos(2\pi f_k \tau) p(\tau)\, d\tau \right), \quad (15)$$

where $p(\tau)$ is the probability density function of $\hat{\tau}$. If $2\pi f_k$ std $\{\hat{\tau}\} << 1$, using the approximation $\cos x \approx 1 - x^2/2$,

$$b\{\hat{X}_k\} \approx \frac{X_k}{2} (2\pi f_k)^2 \,\text{var}\{\hat{\tau}\}. \quad (16)$$

This bias can be eliminated by using the geometric mean of the absolute values of the amplitudes instead of the absolute value of the arithmetic mean of the complex amplitudes [13], since in this procedure the phase error has no effect. The additional standard deviation of the averaged amplitude value due to the randomness of the delay is in the same order as the bias of the arithmetic mean, and decreases proportionally with the reciprocal of the number of averaged records. If $\hat{\tau}$ is normally distributed, it is easy to see that std $\{\hat{X}_k\} \approx \sqrt{2} b\{\hat{X}_k\}$.

It should be mentioned here that individual synchronizations, followed by averaging of the complex amplitudes, do not form an optimal procedure. What should theoretically be done is *ensemble processing of all the data records*, looking for estimates of the true values of the complex amplitudes. However, this method may require a large computer if several data records are processed, and this is undesirable. On the other hand, separate synchronizations and averaging usually give results close to the optimum.

VI. Simulations

The results were checked by simulations. A period of a periodic signal was sampled at $M = 128$ points. Another record was taken with time delay $\tau = 64$ ($\Delta t = 1$). Both records were corrupted by additive zero mean Gaussian white noise, with

$$\mathrm{SNR} = 10 \cdot \log_{10} \left(\frac{\sum_{k=1}^{F} A_k^2/2}{\sigma_n^2} \right) = 3 \text{ dB}. \tag{17}$$

Four methods have been compared:

1. maximum of the interpolated cross-correlation of two measured signals (CC),
2. maximum of the interpolated cross-correlation of filtered signals (bandpass at the signal frequencies before the cross correlation, CCf),
3. frequency domain identification of the delay (ELiS),
4. linear approximation of the nonlinear LS problem (approx).

The delay was estimated from 50 simulations, and the empirical standard deviations were calculated. The 95% confidence bounds of the standard deviations of the estimates, assuming Gaussian distribution, are (0.84 estd $\{\hat{\tau}\}$, 1.24 estd $\{\hat{\tau}\}$).

A. Simulation 1

The signal was a sine wave with $f_1 = 1/128$, $A_1 = 1$. From SNR = 3 dB, $\sigma_n^2 = 0.25$. The lower bound on the standard deviation of the delay was 1.80 from (12). No interpolation was necessary. The results are as follows:

$$\text{estd}\,\{\hat{\tau}_{CC}\} = 3.51, \quad \text{estd}\,\{\hat{\tau}_{CCf}\} = 2.08,$$

$$\text{estd}\,\{\hat{\tau}_{ELiS}\} = 2.07, \quad \text{estd}\,\{\hat{\tau}_{\text{approx}}\} = 2.07.$$

It is obvious that the CC method gives the worst results. The standard deviations of all the other methods are the same, and they do not differ significantly from the theoretical lower bound.

B. Simulation 2

The signal was a multisine with frequencies $1/128 \cdots 15/128$, and amplitudes $A_k = 1$. The SNR was chosen again at 3 dB that is, the variance of the noise was $\sigma_n^2 = 3.76$. The theoretical lower bound is 1.40 from (12). An interpolation factor of 8 was used. The simulation gave the following results:

$$\text{estd}\,\{\hat{\tau}_{\text{cc}}\} = 2.54, \quad \text{estd}\,\{\hat{\tau}_{\text{CCf}}\} = 1.41,$$

$$\text{estd}\,\{\hat{\tau}_{\text{ELiS}}\} = 1.34, \quad \text{estd}\,\{\hat{\tau}_{\text{approx}}\} = 1.42.$$

The last three values indicate good coincidence with the Cramér–Rao lower bound.

VII. Conclusions

An effective method for synchronization of noisy time records has been presented. The estimation of the time delay can be performed using an effective algorithm, which provides better results than those obtained from the commonly used cross-correlation method.

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