Approximate Analysis of the Temperature Induced Stresses and Deformations of Composite Shells

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ABSTRACT: In this paper simple formulas are presented which can be used to estimate the response of composite plates and shells to hygrothermal loads. The layup of the shell can be arbitrary (i.e., it can be symmetric or unsymmetric, balanced or unbalanced).

The formulas serve two purposes. First, they can be used to calculate directly the stresses, strains, and displacements caused by a temperature and a moisture gradient. Second, the formulas can be used to determine the "effective" thermal and moisture expansion coefficients which are the parameters needed in more accurate numerical (FEM) calculations.

The accuracies of the approximate formulas were assessed by sample problems. In these problems the hygrothermal deformations of cylinders and cylindrical segments were calculated by the present approximate formulas and by an exact, three-dimensional analysis. The results of the exact and approximate methods were compared. These comparisons showed that the approximate formulas yield the deformations with a high degree of accuracy.

1. INTRODUCTION

Composite cylinders and cylindrical segments are important structural elements. Their response to a change in temperature (or moisture content) is unexpected; the curvature of cylindrical segments with symmetric layups changes when they are subjected to a constant change in temperature. (This plays the major role in the angle change occurring in the corners of composite laminates during cure, e.g., References [1–6].) This phenomenon is referred to as "springback" (or "spring-forward"). It occurs because of the thickening of the laminate, or

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more precisely, because of the difference between the in-plane and out-of-plane strains in the plies due to the change in temperature and moisture content [1-6].

This effect can be investigated by three-dimensional elasticity solutions [5-11], by three-dimensional finite element methods (e.g., References [2,4]) or by higher order shell theories [12].

Our aim is to determine simple formulas instead of 3-D analysis, based on the classical laminated plate theory [13], to calculate the stresses and strains in composite shells with arbitrary layups subjected to constant and linear temperature and moisture gradients.

Approximate formulas were derived only for cylindrical segments with symmetric layups subjected to a constant change in temperature ([2,3] see Equation (39) of this paper), not for unsymmetric, unbalanced, doubly curved shells.

2. PROBLEM STATEMENT

Consider a composite shell which consists of unidirectional fiber reinforced plies. There is no restriction on the orientation of the plies, the laminate can be unsymmetric and unbalanced. The middle surface of the shell can be flat, cylindrical, or any doubly curved shape (Figure 1). The shell is subjected to linear temperature and moisture gradients through the thickness:

\[ \Delta T = \Delta T_0 + z\Delta T_1, \quad \Delta c = \Delta c_0 + z\Delta c_1 \]

where \( z \) is the coordinate perpendicular to the surface, and \( z \) is equal to zero on the middle surface. The aim is to determine the stresses and deformations in the composite shell.

For plates and cylinders, we will give the expressions of stresses and strains. For general shells only the “effective thermal expansion coefficients” and the “effective moisture expansion coefficients” [13-15] which are necessary inputs in the constitutive relationships for more efficient (analytical or numerical) calculations. The constitutive law is the following:

\[
\begin{pmatrix}
N \\
M
\end{pmatrix}
= \begin{pmatrix}
A & B \\
B & D
\end{pmatrix}
\begin{pmatrix}
\epsilon^e \\
\chi
\end{pmatrix}
- \Delta T_0 \begin{pmatrix}
\epsilon^{\phi}_{T0} \\
\chi_{T0}
\end{pmatrix}
- \Delta T_1 \begin{pmatrix}
\epsilon^{\phi}_{T1} \\
\chi_{T1}
\end{pmatrix}
- \Delta c_0 \begin{pmatrix}
\epsilon^{\phi}_{c0} \\
\chi_{c0}
\end{pmatrix}
- \Delta c_1 \begin{pmatrix}
\epsilon^{\phi}_{c1} \\
\chi_{c1}
\end{pmatrix}
\]

(2)

where \( N \) and \( M \) are the vectors of forces and moments induced in the shell, \( A, \)
B, and D are the stiffness matrices, \( \varepsilon^0 \) and \( \mathbf{x} \) are the vectors of strains and curvature changes of the middle surface. \( \varepsilon^0_0, \mathbf{x}_0 \), and \( \varepsilon^0_1, \mathbf{x}_1 \) are the “effective thermal expansion coefficients” for constant and linear change in temperature; and \( \varepsilon^0_{e0}, \mathbf{x}_{e0} \), and \( \varepsilon^0_{e1}, \mathbf{x}_{e1} \) are the “effective moisture expansion coefficients” for constant and linear change in moisture.

3. METHOD OF SOLUTION

Temperature and moisture affect the deformations of a shell in a similar manner [Equation (2)]. For simplicity hereafter we include only the temperature in the analysis with the understanding that the moisture content can be included in an identical manner as the temperature.

The deformations of a laminated composite shell will be calculated using the thin shell approximation [16]. However, the effect of “thickening” due to the change in temperature will be taken into account by modifying the effective thermal expansion coefficients.

The calculation of deformations and stresses will be evaluated in three steps:

1. First we investigate a plate which has the same layup as the shell in question. We apply a constant change in temperature \( (\Delta T = \Delta T_0 = 1) \) and determine the deformations of the middle surface. These deformations, the in plane strains \( (\varepsilon^0_{T0,p}) \) and the curvatures \( (\mathbf{x}_{T0,p}) \) are the “effective thermal expansion coefficients” of a plate for a constant change in temperature. Then we apply a linear change in temperature \( (\Delta T = z\Delta T_1 = z) \) and determine the deformations of the middle surface. These deformations \( (\varepsilon^0_{T1,p}, \mathbf{x}_{T1,p}) \) are the “effective thermal expansion coefficients” of a plate for a linear change in temperature. Due to the above loads the thickness of the plate may change, which causes no additional stresses.

2. Second, we modify the effective thermal expansion coefficients calculated for a plate taking into account the effect of the curved middle surface. Note that this modification may vary with position on the middle surface depending on the shell geometry.

3. In order to solve for the deformation in a shell with given geometry, loads, and boundary conditions, we have to consider the equilibrium, strain-displacement, and constitutive relationships. The constitutive law is given by Equation (2). To solve these equations we can use any analytical or numerical (FEM) method, which provides the displacements, stresses and strains in the shell. The solution of the aforementioned equation system is not the subject of this paper, but instead we concentrate on the determination of the effective thermal expansion coefficients. However, we will consider some simple cases in which the approximate deformations may readily be obtained: cylinders and cylindrical segments subjected to a change in temperature which varies through the thickness but not along the surface.

We assume that the material behaves in a linearly elastic manner, the deformations are small, and the normals of the undeformed middle surface remain straight and normal to the middle surface after the deformations.
4. EFFECTIVE THERMAL EXPANSION COEFFICIENTS OF COMPOSITE PLATES

The first step, as we mentioned in the previous section, is to determine the deformations of a composite plate due to the change in temperature.

4.1 Notations and Definitions

We use the classical laminated plate theory. The global, off axis, coordinate system is 1,2,3, while the local, on axis coordinate system is x,y,z (Figure 2). The origin of both coordinate systems is on the middle surface. The basic notations and the well-known relationships [13] are listed in Table 1.

In the laminated plate theory, plane stress is assumed. The out-of-plane strain can be calculated as follows [13]:

\[ \varepsilon_z = S_{perp} \sigma^z \]  

(3)

where

\[ S_{perp} = \begin{pmatrix} -\nu_{xx} & -\nu_{xy} & 0 \\ \frac{1}{E_x} & \frac{1}{E_y} & 0 \end{pmatrix} \]  

(4)

The on axis thermal expansion coefficients are denoted by \( \alpha_x, \alpha_y, \alpha_z \).

4.2 Effective Thermal Expansion Coefficients for Constant Change in Temperature

Let an unconstrained plate be subjected to the constant change in temperature: \( \Delta T = \Delta T_s = 1 \). The strains and curvatures of the middle surface due to this loading (denoted by \( \varepsilon_{T0,p} \) and \( \kappa_{T0,p} \)) are the effective thermal expansion coefficients for a constant change in temperature.
Table 1. Stresses and strains in the laminated plate theory. The superscripts \( x \) and 1 refer to the coordinate systems \( x,y,z \) and 1,2,3 respectively. If there is no superscript, the stress or strain is in the global 1,2,3 coordinate system.

**Strains**

\[
\varepsilon^x = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix} \quad \text{or} \quad \varepsilon = \varepsilon^1 = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}
\]

**Stresses**

\[
\sigma^x = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \quad \text{or} \quad \sigma = \sigma^1 = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}
\]

Generalized strains in the middle surface:

\[
\varepsilon^0 = \varepsilon^{01} = \begin{pmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_3^0 \end{pmatrix} \quad \text{and} \quad x = x^1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

\[
\varepsilon = \varepsilon^0 + Z \chi
\]

Generalized stresses in the middle surface:

\[
N = N^1 = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \int_{(h)} \alpha^1 d\tau \quad \text{and} \quad M = M^1 = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \int_{(h)} Z \alpha^1 d\tau
\]

**Constitutive Equations**

\[
(\varepsilon^x - \Delta T \alpha) = S^x \sigma^x \quad \sigma^x = Q^x (\varepsilon^x - \Delta T \alpha)
\]

where

\[
S^x = \begin{pmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_x} & 0 \\
-\frac{\nu_{yx}}{E_x} & \frac{1}{E_y} & 0 \\
0 & 0 & \frac{1}{E_z}
\end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_x \\ \alpha_y \\ 0 \end{pmatrix}, \quad \text{and} \quad Q^x = [S^x]^T
\]

(continued)
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Table 1. (continued).

Constitutive equations for a plate if the change in temperature is zero:

\[
\begin{pmatrix}
N \\
M
\end{pmatrix} = \begin{pmatrix}
A & B \\
B & D
\end{pmatrix} \begin{pmatrix}
\varepsilon_x \\
\varepsilon_y
\end{pmatrix}
\]

A, B, and D are given in Reference [1].

Transformations between stresses, strains, and stiffnesses:

\[
\sigma' = J_{1x} \sigma, \quad \varepsilon' = [J_{1x}^{-1}] \varepsilon, \quad Q' = Q = [J_{1x}^{-1}] Q' [J_{1x}^{-1}]
\]

where

\[
J_{1x} = \begin{pmatrix}
\cos^2 \alpha_{1x} & \sin^2 \alpha_{1x} & 2 \cos \alpha_{1x} \sin \alpha_{1x} \\
\sin^2 \alpha_{1x} & \cos^2 \alpha_{1x} & -2 \cos \alpha_{1x} \sin \alpha_{1x} \\
-\cos \alpha_{1x} \sin \alpha_{1x} & \cos \alpha_{1x} \sin \alpha_{1x} & (\cos^2 \alpha_{1x} - \sin^2 \alpha_{1x})
\end{pmatrix}
\]

and \( J_{1x}^\top \) is the transpose of \( J_{1x} \).

If the deformations (except the thickening) of the plate are constrained, the internal forces are:

\[
N_{T0} = \int_{(h)} [J_{1x}^{-1}] Q' \begin{pmatrix}
-\alpha_x \\
-\alpha_y \\
0
\end{pmatrix} dz, \quad M_{T0} = \int_{(h)} z [J_{1x}^{-1}] Q' \begin{pmatrix}
-\alpha_x \\
0
\end{pmatrix} dz
\]

Applying the opposite of the above forces on the plate produces the same deformations as in an unconstrained plate subjected to the change in temperature:

\[
\begin{pmatrix}
\varepsilon_{T0,P} \\
\chi_{T0,P}
\end{pmatrix} = - \begin{pmatrix}
A & B \\
B & D
\end{pmatrix}^{-1} \begin{pmatrix}
N_{T0} \\
M_{T0}
\end{pmatrix}
\]

The plate changes its thickness due to the temperature change. The thickening contains two effects:

\[
\Delta h_{T0}(z) = \Delta h'_{T0}(z) + \Delta h''_{T0}(z)
\]

where \( \Delta h'_{T0}(z) \) corresponds to the thermal expansion in the z direction, while \( \Delta h''_{T0}(z) \) is induced by the in-plane forces [Equation (3)].

\[
\Delta h'_{T0}(z) = \int_0^z \alpha_z dz, \quad \Delta h''_{T0}(z) = \int_0^z S_{perp} \sigma_{T0} dz
\]
where the in-plane stresses are:

$$\sigma^*_{T_0} = Q^* \left\{ [J^*_x]^{-1}(\epsilon^o_{T_0,\rho} + z \chi_{T_0,\rho}) - \begin{pmatrix} a_x \\ a_y \\ 0 \end{pmatrix} \right\}$$

(9)

4.3 Effective Thermal Expansion Coefficients for Linear Change in Temperature

Let an unconstrained plate be subjected to the linear change in temperature: $$\Delta T = z \Delta T_1 = z$$. The strains and curvatures of the middle surface due to this load (denoted by $$\epsilon^o_{T_1,\rho}$$ and $$\chi_{T_1,\rho}$$) are the effective thermal expansion coefficients for a linear change in temperature.

If we constrain the deformations of the plate (except the thickening) the internal forces are:

$$N_{T_1} = \int_{(h)} z [J_{1x}]^{-1} Q^* \begin{pmatrix} \alpha_x \\ -\alpha_y \\ 0 \end{pmatrix} \text{d}z,$$

$$M_{T_1} = \int_{(h)} z^2 [J_{1x}]^{-1} Q^* \begin{pmatrix} -\alpha_x \\ \alpha_y \\ 0 \end{pmatrix} \text{d}z$$

(10)

Applying the opposite of the above forces on the plate produces the same deformations as in an unconstrained plate subjected to the change in temperature:

$$\left( \begin{array}{c} \epsilon^o_{T_1,\rho} \\ \chi_{T_1,\rho} \end{array} \right) = -\left( \begin{array}{cc} A & B \\ B & D \end{array} \right)^{-1} \left[ \begin{array}{c} N_{T_1} \\ M_{T_1} \end{array} \right]$$

(11)

The change in thickness is:

$$\Delta h_{T_1}(z) = \Delta h'_{T_1}(z) + \Delta h''_{T_1}(z)$$

(12)

where

$$\Delta h'_{T_1}(z) = \int_0^z z \alpha_x \text{d}z,$$

$$\Delta h''_{T_1}(z) = \int_0^z S_{\text{perp}} \sigma^*_{T_1} \text{d}z$$

(13)

with

$$\sigma^*_{T_1} = Q^* \left\{ [J^*_x]^{-1}(\epsilon^o_{T_1,\rho} + z \chi_{T_1,\rho}) - \begin{pmatrix} a_x \\ a_y \\ 0 \end{pmatrix} \right\}$$

(14)

The derived $$\epsilon^o_{T_0,\rho}, \epsilon^o_{T_1,\rho}, \chi_{T_0,\rho}, \chi_{T_1,\rho}$$ are the effective thermal expansion coefficients of a plate. If we apply the temperature change $$\Delta T = \Delta T_0 + z \Delta T_1$$ on a plate with free edges, the deformations of the middle surface are

$$\epsilon^o = \Delta T_0 \epsilon^o_{T_0,\rho} + \Delta T_1 \epsilon^o_{T_1,\rho}, \quad \chi = \Delta T_0 \chi_{T_0,\rho} + \Delta T_1 \chi_{T_1,\rho}$$

(15)
However, if the middle surface is curved, we have to modify these deformations as it will be shown in the next section.

5. THERMAL EXPANSION COEFFICIENTS OF COMPOSITE SHELLS

In this section we will investigate the stresses and deformations which must be added to those in a plate, caused by the thickening of the shell and by the stretching of the middle surface. First we introduce a new \( \xi, \eta, \zeta \) coordinate system, where \( \zeta \) is perpendicular to the surface, \( \xi, \eta \) are in the planes of the principal curvatures. The radii of curvatures of the shell in the \( \xi, \zeta \) and in the \( \eta, \zeta \) planes are denoted by \( R_\xi \) and \( R_\eta \).

Let us transform the effective thermal expansion coefficients into the \( \xi, \eta, \zeta \) coordinate system:

\[
\varepsilon_{T,p}^{\xi} = [J_{\xi,\xi}^{-1}] \varepsilon_{T,p}^{\eta}, \quad \varepsilon_{T,p}^{\eta} = [J_{\eta,\eta}^{-1}] \varepsilon_{T,p}^{\eta}
\]

(16)

In this section we apply the subscript \( T \) instead of \( T_0 \) or \( T_1 \), with the understanding that the expressions are valid for both the constant and the linear temperature distribution.

5.1 Effect of Thickening of the Shell

Due to the thickening, \( \varepsilon_\xi \) and \( \varepsilon_\eta \) are induced in the laminate. (The induced \( \varepsilon_\xi \) is shown in Figure 3.) Assuming that \( z \ll R_\xi \) and \( z \ll R_\eta \), the strains are:

\[
\varepsilon_{\xi,\eta} = \frac{\Delta h_T(z)}{R_\xi}, \quad \varepsilon_{\eta,\eta} = \frac{\Delta h_T(z)}{R_\eta}
\]

(17)

where \( \Delta h_T(z) \) is the thickening of the laminate [Equations (7) or (12)]. If we con-

![Figure 3. Illustration of \( \varepsilon_\xi \) caused by the "thickening." ](image)
strain the deformations the internal stresses are as follows:

\[
N_h = \int_{(h)} Q[J]_{1} \begin{pmatrix} \varepsilon_{\xi, h} \\ 0 \end{pmatrix} dz, \quad M_h = \int_{(h)} zQ[J]_{1} \begin{pmatrix} \varepsilon_{\eta, h} \\ 0 \end{pmatrix} dz
\] (18)

Applying the opposite of these forces, the additional deformations of the middle surface due to the thickening can be calculated:

\[
\begin{pmatrix} \varepsilon_{\xi, T, h} \\ x_{T, h} \end{pmatrix} = - \begin{pmatrix} A & B \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} N_h \\ M_h \end{pmatrix}
\] (19)

5.2 Effect of Stretching of the Middle Surface of the Shell

In the laminated plate theory the strains (\(\epsilon\)) in a plate can be calculated from the deformations (\(\varepsilon^o\) and \(x\)) of the middle surface:

\[
\epsilon = \varepsilon^o + z x
\]

or in the \(\xi, \eta, \zeta\) coordinate system:

\[
\epsilon_\xi = \varepsilon_\xi^o + z \chi_\xi, \quad \epsilon_\eta = \varepsilon_\eta^o + z \chi_\eta, \quad \gamma_\eta = \varepsilon_\eta^o + z \chi_\eta
\] (20)

These are also the usual approximations of thin shell theory. However, in a shell the strains are not linear through the thickness, we can approximate them as [16,17]

\[
\epsilon_\xi = (\varepsilon_\xi^o + z \chi_\xi) \left(1 - \frac{z}{R_\xi}\right), \quad \epsilon_\eta = (\varepsilon_\eta^o + z \chi_\eta) \left(1 - \frac{z}{R_\eta}\right)
\]

\[
\gamma_{\eta \xi} = \varepsilon_{\eta \xi}^o + z \chi_{\eta \xi}
\] (21)

Figure 4 illustrates that the constant strain \(\epsilon_\xi\) is caused by stretching together with a change in curvature of the middle surface. (Note that different authors use different definitions for the curvature. Flügge [16] defines it as the mathematical change in curvature of the middle surface. Timoshenko [17] defines the curvature as the relative angular displacement of the two opposite edges of a shell element over the element's length. The latter is the definition that we use in this paper.)

If the applied shell theory makes use of Equation (20) instead of Equation (21), significant error may occur in the deformations caused by the change in temperature. We can eliminate this error by changing the effective thermal expansion coefficients. The difference between the strains in a plate and in a shell due to the deformations of the middle surface of the shell, \(\varepsilon_{T, p}^o\) and \(x_{T, p}^o\) are Equations (20) and (21):

\[
\varepsilon_{\xi, s} = - \frac{z}{R_\xi} (\varepsilon_{\xi T, p}^o + z \chi_{\xi T, p}), \quad \varepsilon_{\eta, s} = - \frac{z}{R_\eta} (\varepsilon_{\eta T, p}^o + z \chi_{\eta T, p})
\] (22)
where $\varepsilon_{T,P}^o$ and $\varepsilon_{T,P}^e$ are the first two elements of $\varepsilon_{T,P}^e$ and $\kappa_{T,P}^e$ and $\kappa_{T,P}^o$ are the first two elements of $\kappa_{T,P}^e$ [Equation (21)]. To get the deformations of the middle surface we have to follow the same steps as in the case of the thickening. If we constrain the deformations the internal stresses are as follows:

$$N_s = \int_{(h)} Q[J'_t] \begin{pmatrix} \varepsilon_{T,s}^o & 0 \\ \varepsilon_{q,s}^o & 0 \end{pmatrix} \frac{dZ}{z}, \quad M_s = \int_{(h)} zQ[J'_t] \begin{pmatrix} \varepsilon_{T,s}^o & 0 \\ \varepsilon_{q,s}^o & 0 \end{pmatrix} dz \quad (23)$$

Applying the opposite of these forces, the additional deformations caused by the stretching of the middle surface can be calculated:

$$\left( \begin{array}{c} \varepsilon_{T,s}^o \\ \kappa_{T,s}^o \end{array} \right) = -A^{-1}B \begin{pmatrix} N_s \\ M_s \end{pmatrix} \quad (24)$$

5.3 The Effective Thermal Expansion Coefficients

To get the effective thermal expansion coefficients for a shell we have to modify those of a plate. For constant or linear change in temperature:

$$\left( \begin{array}{c} \varepsilon_{T}^o \\ \kappa_{T}^o \end{array} \right) = \left( \begin{array}{c} \varepsilon_{T,P}^o \\ \kappa_{T,P}^o \end{array} \right) + \left( \begin{array}{c} \varepsilon_{T,h}^o \\ \kappa_{T,h}^o \end{array} \right) + \left( \begin{array}{c} \varepsilon_{T,s}^o \\ \kappa_{T,s}^o \end{array} \right) \quad (25)$$

The terms in Equation (25) are defined by Equations (6), (11), (19) and (24).
5.4 Stresses in a Ply

In the calculation of stresses in a shell we have to take into account the thickening and the stretching of the middle surface. Hence the stresses are:

$$\sigma = Q \left\{ \begin{array}{c}
\epsilon' + z \chi - \Delta T_i J_{i_1} \left( \begin{array}{c} \alpha_i \\ \alpha_i \\ 0 \\ \end{array} \right) - \Delta T_i J_{i_1}' \\
\frac{1}{R_t} \left[ -\Delta h_{T_0}(z) + z(\epsilon_{T_0} + z\chi_{T_0}) \right] \\
\frac{1}{R_n} \left[ -\Delta h_{T_1}(z) + z(\epsilon_{T_1} + z\chi_{T_1}) \right] \\
\end{array} \right. \\
\right\}
$$

where $\epsilon'$ and $\chi$ are the deformations of the middle surface.

Let us emphasize again that the terms due to the "stretching" are necessary only if shell theory defines the curvature due to Reference [17] and make use of Equation (20).

5.5 Approximate Calculation of the Thermal Expansion Coefficients for Constant Change in Temperature

We can further simplify the expressions derived in the previous sections for constant change in temperature approximating the displacement $\Delta h(z)$ [Equation (7)] by a linear function:

$$\Delta h(z) = z\alpha_{s, p}$$

and neglecting the second (quadratic) terms in Equation (22)

$$\epsilon_{t,s} = -\frac{z}{R_t} \epsilon_{tT, p}, \quad \epsilon_{n,s} = -\frac{z}{R_n} \epsilon_{nT, p}$$

(28)
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In Equation (27) \( \alpha_{z,p} \) is the average thermal expansion coefficient of a laminate perpendicular to the surface, which is calculated as follows:

\[
\alpha_{z,p} = \frac{1}{h} \left[ \Delta h_{T_0} \left( \frac{h}{2} \right) - \Delta h_{T_0} \left( -\frac{h}{2} \right) \right]
\]  

(29)

This “thickness expansion coefficient” for a symmetric laminate was first derived by Pagano [18].

Applying the above approximation in Equations (25) we obtain:

\[
\begin{pmatrix}
\varepsilon_{T_0}^z \\
\chi_{T_0}^z
\end{pmatrix}
= \begin{pmatrix}
\varepsilon_{T_0}^{01, p} \\
\chi_{T_0}^{1, p}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 \\
1/R_\xi \left( \varepsilon_{z,T_0,p}^{\xi} - \alpha_{z,p} \right) & 0 & J_{\xi}^z \left( \varepsilon_{z,T_0,p}^{\xi} - \alpha_{z,p} \right)
\end{pmatrix}
\]  

(30)

and the internal stresses from Equation (26) are:

\[
\sigma^i = Q^i \left\{ \left( \varepsilon^{0} + \varepsilon x \right) - \Delta T_\varepsilon \chi_{T_0}^i \left( \alpha_x \right) - \Delta T_0 J_{\xi}^z \varepsilon^{0} - \varepsilon \chi_{T_0}^i \left( \alpha_x \right) \right\}
\]  

(31)

These simplified expressions are applicable if the out-of-plane thermal expansion coefficients of the plies in a laminate are close to each other, which is true if hybrid composites are not considered.

Note, if the in-plane and out-of-plane deformations of a plate caused by the temperature change are the same \( \varepsilon_{T_0}^{\varepsilon} = \varepsilon_{T_0}^{\varepsilon}, \alpha_{z,T_0,p} = \alpha_{z,T_0,p} \), the last terms in Equation (30) and in Equation (31) are zero. This is the case for the isotropic materials.

6. COMPOSITE CYLINDERS

Consider a cylinder with the radius \( R \) and axis along the \( 1 \) direction. The previous expressions can be simplified in this case, taking into account that \( 1 \equiv \xi, 1/R_\xi = 0, R_\eta = R, \) and \( J_{\xi} \) is a unit matrix.
The approximate expression for a constant change in temperature [Equation (30)] is:

\[
\left( \frac{\varepsilon_{\theta \theta}}{\kappa_{T_0}} \right) = \left( \frac{\varepsilon_{\theta \theta}}{\kappa_{T_0, \rho}} \right) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{R} (\varepsilon_{\theta \rho}^z - \alpha_{\theta, \rho}) & 0 & 0 & 0 \end{pmatrix}
\]

(32)

and the internal stresses are:

\[
\sigma^t = Q^t \begin{pmatrix} 0 \\ \kappa(z \mathbf{\kappa}) - \Delta T^t \mathbf{J}_{zz} \alpha_{zz} + \Delta T^t \mathbf{z} \begin{pmatrix} \alpha_{zz} \\ \alpha_{zz} \\ 0 \end{pmatrix} + \frac{1}{R} (\varepsilon_{\theta \rho}^z - \alpha_{\theta, \rho}) \end{pmatrix} \]

(33)

The third step in the solution (see “Method of Solution” Section) is to determine the stresses and deformations in the shell due to the applied loads considering the equilibrium equations and the relationships between the strains and displacements.

The displacements of the middle surface in the axial \( x_1 \), circumferential \( \theta \) (or \( x_2 \)), and radial \( r \) (or \( x_3 \)) directions are denoted by \( u \), \( v \), and \( w \), respectively. The relationships between the displacements and the deformations of the middle surface of the cylinder are [17]:

\[
\begin{align*}
\varepsilon_1 &= \frac{\partial u}{\partial x_1}, \\
\varepsilon_2 &= \frac{\partial v}{R \partial \theta} - \frac{w}{R}, \\
\varepsilon_6 &= \frac{\partial u}{R \partial \theta} + \frac{\partial v}{\partial x} \\
x_1 &= -\frac{\partial^2 w}{\partial x_1^2}, \\
x_2 &= \frac{1}{R^2} \left( \frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2} \right), \\
x_6 &= \frac{1}{R} \left( \frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x \partial \theta} \right)
\end{align*}
\]

(34)

If the strains are constant in the cylinder, \( x_1 \) must be zero and the expressions for the displacements from Equation (34) are as follows:

\[
\begin{align*}
u &= x_1 \varepsilon_1 + \theta R (\varepsilon_6 - R x_6) \\
v &= \theta R^2 x_2 + x R x_6 \\
w &= R (\varepsilon_2 - R x_2)
\end{align*}
\]

(35)
6.1 Cylindrical Segments

Consider a cylindrical segment subjected to a change in temperature which does not vary on the surface. In this case, the deformations of the middle surface \((e^o, x)\) and the forces \((N, M)\) are constant in the shell (except in the neighborhood of the curved boundaries). Hence we can determine the twelve elements of \((e^o, x, N, M)\) from the following twelve equations:

\[
\begin{pmatrix}
N \\
M
\end{pmatrix} = 
\begin{pmatrix}
A & B \\
B & D
\end{pmatrix}
\begin{pmatrix}
e^o \\
x
\end{pmatrix} - 
\Delta T_o \begin{pmatrix}
e^o_{r_0} \\
x_{r_0}
\end{pmatrix} - 
\Delta T_1 \begin{pmatrix}
e^o_{r_1} \\
x_{r_1}
\end{pmatrix}
\]

(36)

\[N_1 = 0, \quad N_2 = 0, \quad N_6 = 0, \quad M_2 = 0, \quad M_6 = 0, \quad x_1 = 0\]

(37)

Knowing \(e^o\) and \(x\) the stresses in the segment can be calculated from Equation (26).

The springback (Figure 5) of a segment is

\[
S = \frac{\delta}{\gamma} = \frac{1}{\gamma} \int_0^\gamma \left( \frac{\partial y}{R \partial \theta} - \frac{\partial^2 w}{R \partial \theta^2} \right) d\theta = x_2 R
\]

(38)

where \(x_2\) is the second element of \(x\), \(\gamma\) is the angle of the segment, and \(\delta\) is the change in angle.

Figure 5. Illustration of springback of an open cylinder.
If a cylindrical segment with symmetric layup is subjected to a constant change in temperature, $\kappa_{r,0,p} = 0$ and the calculation of springback becomes even simpler. From Equations (32) and (36)-(38):

$$S = \Delta T_c (\epsilon_2, r_0, p - \alpha_z)$$  \hspace{1cm} (39)

This expression is identical to the formula of References [2] and [3].

6.2 Closed Cylinders

Consider a closed cylinder subjected to a change in temperature which does not vary on the surface. The deformations of the middle surface $(\epsilon^0, \kappa)$ and the forces $(N, M)$ are constant in the shell (except in the neighborhood of the curved boundaries), hence the displacements are described by Equation (35), and

$$\kappa_1 = 0$$  \hspace{1cm} (40)

The displacements must be periodic functions of $\theta$, therefore from Equation (35):

$$\kappa_2 = 0, \quad \epsilon_0 - R\kappa_6 = 0$$  \hspace{1cm} (41)

The cylinder is neither loaded by an axial force, nor by an internal pressure, nor by a torque, hence from the equilibrium equations [17]:

$$N_1 = 0, \quad N_2 = 0, \quad RN_6 + M_6 = 0$$  \hspace{1cm} (42)

The twelve unknowns in $\epsilon^0$ and $\kappa$ can be determined from Equations (36) and (40)-(42). Knowing the strains, the stresses in the cylinder can be calculated from Equation (26).

7. SAMPLE PROBLEMS

In order to investigate the accuracy of the derived approximate formulas sample problems were calculated for cylinders and cylindrical segments with different layups and geometries. These results are designated as “APPROXIMATE.” The calculated deformations and stresses were compared to the “EXACT” solutions, obtained by the “Segment” code [11]. We compared the results also to those obtained by further simplifications, calculating the deformations and stresses without taking into account the effect of stretching and thickening. These calculations are referred to as “PLATE” solutions.

In every example the shell is made out of E-glass epoxy composite. The material properties are listed in Table 2.

7.1 Cylinders and Cylindrical Segments Subjected to Constant Change in Temperature

In the following examples the laminates are subjected to a constant change in temperature, $\Delta T = 1^\circ C$ at both the inner and the outer surfaces. The approximate formulas given by Equations (32) and (33) were used.
Table 2. Material properties of E-glass/epoxy composites [1].

Stiffness Parameters
- longitudinal Young's modulus \( E_x = 38.6 \) GPa
- transverse in-plane Young's modulus \( E_y = 8.27 \) GPa
- transverse out-of-plane Young's modulus \( E_z = 8.27 \) GPa
- shear moduli \( G_{xy} = G_{yz} = 4.14 \) GPa
- \( G_{xz} = 2.95 \) GPa
- Poisson's ratios \( \nu_{yz} = 0.4 \), \( \nu_{zx} = \nu_{xy} = 0.26 \)

Thermal Expansion Coefficients
- longitudinal \( \alpha_x = 8.6 \times 10^{-6}/\degree\text{C} \)
- transverse \( \alpha_y = \alpha_z = 22.1 \times 10^{-6}/\degree\text{C} \)

First, symmetric, cross ply cylindrical segments were investigated. The data are plotted as a function of the percentage of zero direction plies. (The zero direction is parallel to the axis of the cylinder.) In the approximate calculation for symmetric laminates, the layup has no effect on the deformations; only the ratio of the zero to ninety direction plies affects the effective thermal expansion coefficients (see the top of Figure 6). The springback [Equation (38)] of the segments are shown in the bottom of Figure 6. The approximate value is the difference between the thermal expansion coefficients plotted at the top of the figure. Using the plate solution, the calculated springback is zero. The exact calculations were performed for two extreme cases, first, if all the zero direction plies are in the middle of the laminate, second, if all the zero plies are at the two outer surfaces of the laminate.

The springback of symmetric angle ply laminates is investigated in Figure 7. Figure 8 shows the effect of the radius/thickness ratio on the springback for a symmetric cross ply segment.

Unsymmetric cross ply laminates were investigated in Figure 9.

In Figures 10 and 11 the stresses for a cylindrical segment and for a closed cylinder are given. In both cases the ratio of the radius to the thickness is ten and the layup is \([45/-45/0/90_{2}/0/-45/45]\).

In all the above cases we found good agreement between the exact and approximate calculations. (\( \sigma_3 \) is not predictable by the approximate calculation, but this stress is much smaller than \( \sigma_1 \) and \( \sigma_2 \).)

7.2 Cylindrical Segments Subjected to Linear Change in Temperature

In the following examples symmetric composite cylindrical segments are considered. The radius is 0.1 m and the thickness is 0.01 m in every case. The segments are subjected to linear change in temperature, \( \Delta T = z \), the inner surface is cooled down by 0.005°C, the outer is heated up by 0.005°C.

Figure 12 shows the calculated springback of cross ply segments. There is
Figure 6. Investigation of a \([0_{p/2}/90_{50-p/2}]_5\) and a \([90_{50-p/2}/0_{p/2}]_5\) laminate. Top: effective thermal expansion coefficients; bottom: springback due to the approximate and exact calculation.
Figure 7. Investigation of a \([\phi/ - \phi]\) laminate. Top: effective thermal expansion coefficients; bottom: springback due to the approximate and exact calculation.
Figure 8. Investigation of the effect of thickness on the accuracy of springback of a cross ply \([90/0/90/0]_s\) laminate.

Figure 9. Calculation of springback on an unsymmetric, \([0/90]_p\) laminate.
Figure 10. Stresses (kPa) in a \([45/-45/0/90]_S\) cylindrical segment subjected to \(\Delta T = \Delta T_0 = 1^\circ C\) change in temperature.

Figure 11. Stresses (kPa) in a \([45/-45/0/90]_S\) cylinder subjected to \(\Delta T = \Delta T_0 = 1^\circ C\) change in temperature.
practically no difference between the results obtained by the exact, approximate and plate calculations.

The stresses of a [45/−45/0/90] cylindrical segment subjected to a linear change in temperature are given in Figure 13. The stresses (except $\sigma_3$) due to the exact and approximate calculation are practically identical, the results from the plate solution are also very close.

**Figure 12.** Investigation of the springback of $[0_{p/2}/90_{100-p/2}]_5$ and of $[90_{100-p/2}/0_{p/2}]_5$ laminates subjected to linear change in temperature. The results from the exact, approximate, and plate calculations are on the top of each other.

**Figure 13.** Stresses (kPa) in a [45/−45/0/90] cylindrical segment subjected to linear change in temperature.
Figure 14. Stresses (kPa) in a unidirectional [0] cylindrical segment subjected to linear change in temperature.

The hoop stresses calculated by the exact (or by the approximate) and by the plate solution are very different for unidirectional, [0] composites (Figure 14). However, these stresses are relatively small compared to the axial stresses.

8. CONCLUSIONS

Approximate formulas were derived to determine the stresses and deformations of composite shells subjected to temperature loads.

The accuracy of the formulas was investigated by calculating several numerical examples by the approximate expressions and by the "exact" solutions. These comparisons showed that the approximate formulas yield the deformations with a high degree of accuracy in every investigated case ($R/h > 10$).

The effect of "thickening" and "stretching" plays an important role in the deformations of shells subjected to constant change in temperature.

The "thickening" and "stretching" have very small effect on the deformations of shells subjected to linear change in temperature.

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