The BER Performance of OFDM Systems using Non-Synchronized Sampling

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Abstract — In fully digital receivers, carrier and timing information is derived from samples of the (anti-aliasing-filtered) received continuous-time signal. In case of synchronized sampling, this information is used to align the sampling clock of the receiver with the remote transmit clock. In non-synchronized sampling systems, the sampling at the receiver is performed by means of a fixed free-running clock, and additional post-processing is necessary to perform timing correction in the digital domain. In this paper, we investigate the effect of non-synchronized sampling on the BER performance of OFDM systems. We calculate the BER degradation caused by a given frequency offset between receiver and transmitter clock, as compared with the case of ideal sampling. The obtained results are compared with the performance of synchronized sampling systems.

I. INTRODUCTION

In a digital OFDM (Orthogonal Frequency Division Multiplexing) receiver, the received continuous-time signal is sampled at instants determined by the receiver clock. At the receiver, demodulation of the OFDM symbol consists of an FFT (Fast Fourier Transform) operation performed on a sequence of samples from the received OFDM symbol.

In synchronized sampling systems the timing algorithm controls a VCXO (Voltage Controlled Crystal Oscillator) in order to align the receiver clock with the transmitter clock [1]. In case of non-synchronized sampling, the sampling rate remains fixed. Therefore, an offset between receiver and transmitter clock rate will result in additional distortion of the signal at the output of the FFT performed by the receiver. The effect of a clock frequency offset is twofold. First, the useful signal component is rotated and attenuated. The rotation of the received symbol is compensated for in the digital domain by a frequency domain equalizer (ROTOR). In addition, the impairment gives rise to Inter Carrier Interference (ICI), i.e. signal components caused by carriers other than the considered carrier. Figure 1 depicts the difference between the two synchronizing structures. In this figure, \( H(f) \) represents the transfer function of the transmission channel and \( r(t) \) is the (continuous-time) signal at the input of the receiver.

In section 2, we calculate the received signal at the output of the FFT, given a frequency offset between transmitter and receiver clock. In section 3, we compute for a given offset, the degradation of the BER, expressed in dB. Finally, in section 4, we compare the performance of systems using synchronized and non-synchronized sampling.

Fig. 1a. OFDM receiver with VCXO (synchronized sampling)

Fig. 1b. OFDM receiver with fixed Crystal (non-synchronized sampling)
II. THE OFDM RECEIVER

Essentially, an OFDM signal is the sum of a large number of QAM (or QPSK) modulated carriers (sub-channels). The carrier spacing equals the OFDM symbol rate 1/T [2,3]. OFDM also has been referred to as discrete multitone (DMT) [4]. In case the transmission channel characteristic is slowly varying in time, the size of the QAM constellations can be chosen according to the signal-to-noise ratio (SNR) at each of the sub-channels, in order to obtain the same BER performance for each of the QAM sub-channels [2,3].

During the m-th symbol period of duration T, the complex envelope of the transmitted OFDM signal can be expressed as

\[ s_m(t) = \sum_{k=0}^{N-1} a_{m}^k e^{j2\pi k t T} \]

where \( a_{m}^k \) denotes the m-th transmitted symbol on the carrier k (at carrier frequency k/T) and N is the total number of carriers. In a practical system, samples of the OFDM symbol (1) are generated by means of an N-point IFFT on the symbols \( a_{m}^k \), k \( \in \{0, N-1\} \) [5].

At the receiver, the signal \( r(t) = \sum_{m=-\infty}^{+\infty} s_m(t-mT) + n(t) \) (2)
is sampled at a sampling rate \( f_s+\Delta f \), where \( n(t) \) denotes the (filtered) additive white Gaussian noise (AWGN), \( f_s = N/T \) and \( \Delta f \) is the sampling frequency offset between receiver and transmitter. The receiver performs an FFT on consecutive blocks of N samples, in order to detect the OFDM symbols.

Because of the frequency mismatch, the OFDM symbol duration at the receiver, \( N/(f_s+\Delta f) \), differs from the one at the transmitter, \( N/f_s \). Hence, symbol synchronization must be performed by a timing algorithm. The algorithm ensures that the block of N samples is well aligned within the corresponding symbol period. This requires that, at regular intervals, samples are robbed (if \( \Delta f > 0 \)) or stuffed (\( \Delta f < 0 \)). Let us define \( \delta_k \) as the number of samples of the received sample sequence to be robbed or stuffed during the reception of the k-th OFDM symbol. For each OFDM symbol k, \( \delta_k \) can take the values \{-1,0,1\}. The N consecutive samples belonging to the m-th symbol that are fed to the FFT can be expressed as

\[ r_m^n = r\left(\frac{mN+n+\delta}{f_s+\Delta f}\right) \quad n \in \{0, N-1\} \]

where \( \delta = \sum_{k=0}^{m-1} \delta_k \) (4)

Defining

\[ a_m^k = a_m^k e^{j2\pi k (mN+n)/(f_s+\Delta f)} \]

we can rewrite (3) as

\[ r_m^n = \sum_{k=0}^{N-1} a_{m}^k e^{j2\pi k n N/(f_s+\Delta f)} + n_m^n \]

where \( n_m^n \) denotes the AWGN contribution.

In reality, the transmitted signal will be distorted during transmission over the channel. To avoid Inter Carrier Interference (ICI) and Inter Symbol Interference (ISI) at the receiver, each symbol is extended with a cyclic prefix (i.e. a repetition of the last samples) before transmission [6]. At the receiver, the prefix is removed before FFT. It can be shown [6] that the use of a cyclic prefix ensures orthogonality between the carriers and the results obtained hereafter remain valid if in (6) \( a_m^k \) is replaced by \( a_m^k H_k \), where \( H_k \) is the FFT of the channel impulse response. The guard time combined with an appropriate symbol timing algorithm guarantees that all samples \( r_m^n \) (\( n \in \{0, N-1\} \)) belong to the signal transmitted during the m-th symbol period. As a result, no ISI will appear. However, in order to keep the expressions tractable, the notations do not reflect the effect of a guard time.

The FFT operates on \( \{r_m^n\} \), \( n \in \{0, N-1\} \), to produce \( R_m^n \). After some elementary calculations, one obtains (without considering the AWGN contribution),

\[ R_m^n = \hat{a}_m^n - \sum_{k=0}^{N-1} a_{m}^k I_{k,n} \]

where

\[ I_{k,n} = \frac{1}{N} \frac{\sin(\pi(k-f_s+n)/f_s+\Delta f))}{\sin(\pi(k-f_s+n)/f_s+\Delta f))} e^{j\pi N^{-1}(k-f_s+n)/(f_s+\Delta f)} \]

We observe that each output of the FFT, \( R_m^n \), consists of several contributions:
- The first term in (7) denotes the useful component: the transmitted symbol, \( a_m^n \), is rotated and attenuated. Both effects depend on the considered carrier frequency \( n/f_s \). The angle over which the symbol \( a_m^n \) is rotated equals:

\[ \theta_m^n = 2\pi N^n [mN+n]/f_s+\Delta f + \arg [I_{m,n}] \]

Remark that the angle \( \theta_m^n \) is proportional to the tone index \( n \) and increases linearly for successive OFDM symbols as...
long as no samples are robbed or stuffed ($\delta = 0$). Consequently, the received points of a QAM constellation are rotating at a velocity depending linearly on the carrier index. The rotation is compensated for by means of a frequency domain equalizer (ROTOR). The equalizer consists of one complex-valued tap for each FFT output and is updated every OFDM symbol. Robbing or stuffing a sample ($\delta_m = \pm 1$) results into an additional rotation over an angle $(\Delta \theta_m)$ equal to

$$\Delta \theta_m = e^{j2\pi \delta_m \frac{n_f}{N_f} \Delta f}$$

The ROTOR values should be corrected accordingly. The useful component is attenuated by a factor $l_{in}$.

$$D_n = 10 \log_{10} \left( 1 + K_n \frac{E_s}{N_0} \left( \frac{\Delta f^2}{f_s} \right) \right) \quad (9)$$

where

$$K_n = \frac{1}{N_s^2} \sum_{k=0}^{N_s-1} \pi^2 k^2 \sin^2 \left( \frac{\pi(k-n)}{N} \right) \quad (10)$$

IV. RESULTS AND DISCUSSION

In a non-synchronized OFDM system, for a sufficiently large $E_s/N_0$, we see from (8) that the degradation, $D_n$, for the carrier $n/T$ mainly results from ICI.

![Graph](image)

**Fig. 2.** $K_n$ as function of the carrier index $n$

In order to evaluate $D_n$, figure 2 depicts $K_n$ as a function of the frequency index $n$. The total number of carriers ($N$) was chosen to be 256. We observe that over a large area, $K_n$ is proportional to the square of the carrier index. Hence, if the argument of the logarithmic function in (9) is close to 1, the degradation is proportional to the square of the relative frequency offset and to the square of the carrier index. We can approximate expression (9) by (appendix B)

$$D_n = 10 \log_{10} \left( 1 + \frac{E_s \Delta f^2}{3 N_0} \cdot \frac{\pi n \Delta f}{f_s} \right) \quad (11)$$

except for the smallest and highest carrier indexes.

In figure 3, we depict the degradation as a function of the relative frequency offset (expressed in ppm) for the tone (249/T) suffering the most from ICI. The OFDM signal consists of 256 modulated carriers. Figure 4 depicts, for different values of the relative frequency offset, the maximum number of bits per symbol to be transmitted on sub-carrier 249/T as function of the $E_s/N_0$ in order to obtain a BER $10^{-7}$ [6].
Fig. 3. Degradation as function of the frequency offset (ppm)

In applications such as high speed transmission over the unshielded twisted pair wire (ADSL and HDSL), the SNR at each of the sub-carriers is known by the transmitter. The size of the QAM constellation modulating each carrier frequency is determined depending on the SNR at the considered carrier. This ensures the same BER performance for each received QAM signal. The degradation due to a sampling frequency error for the n-th carrier ($D_n$) can be expressed by (8) where $N_0$ is substituted by the power density of the noise at carrier $n/T$.

In an OFDM system with synchronized sampling, information on the instantaneous frequency offset between receiver and transmitter clock is derived from the rotation of the received constellation points. This information is used by the DPLL (Digital Phase Locked Loop) to control the VCXO that determines the sampling instants. This assures that the average frequency offset between receiver and transmitter clock is zero. The degradation $D_n$ then depends on the variance of the instantaneous sampling frequency offset which is determined by the closed loop bandwidth of the DPLL.

The received constellation points which serve as input to the DPLL are obtained at the output of the FFT. As the FFT operates on a symbol per symbol basis, the VCXO is updated only once every OFDM symbol period. Hence, the instantaneous frequency offset remains fixed during each OFDM symbol. Therefore, the degradation can be expressed by (9) where $(\Delta f)^2$ is replaced by $E[(\Delta f)^2]$.

V. CONCLUSION

We conclude that non-synchronous sampling systems are much more sensitive to a frequency offset between transmitter and receiver clock, compared with synchronous sampling systems. For non-synchronous sampling systems, we showed that the degradation due to a frequency sampling offset depends on the square of the carrier index and on the square of the relative frequency offset.

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REFERENCES


APPENDIX A

The variance of the ICI can be expressed as

\[
\text{var}[IC1] = E_n \left[ \sum_{k=0}^{N-1} x_k^2 \right] - E_n \left[ x_n \right]^2
\]

For small \(\Delta f/f_s\), this expression can be approximated by

\[
\text{var}[IC1] = E_n \left[ \sum_{k=0}^{N-1} \sin^2 \left( \frac{\pi (x(k-n) + \Delta f)}{N} \right) \right]
\]

One observes that the variance depends on the carrier index \(n\).

APPENDIX B

Expression (11) can be obtained from (8) (for |H_0|=1) considering that

\[
\sum_{k=0}^{N-1} \left| I_{k,n} \right|^2 = \sum_{k=0}^{N-1} \left| I_{k,n} \right|^2 - \left| l_{n,n} \right|^2
\]

The first term of this expression can be written as

\[
\sum_{k=0}^{N-1} \left| I_{k,n} \right|^2 = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{v=0}^{N-1} e^{j2\pi N (x_n f_s + \Delta f)} e^{j2\pi N (x_n f_s + \Delta f)}
\]

where \( \theta = \frac{f_s}{f_s + \Delta f} \)

For a large range of values of \(n\), \(n \in [0, N-1]\), and \(0 = 1\), the above expression becomes

\[
\sum_{k=0}^{N-1} \left| I_{k,n} \right|^2 = 1
\]

The second term can be expressed, using Taylor expansion, as

\[
\left| l_{n,n} \right|^2 = \frac{1}{N^2} \left[ \frac{\sin \left( \frac{\pi (x_n f_s + \Delta f)}{N} \right)}{\sin \left( \frac{\pi (x_n f_s + \Delta f)}{N} \right)} \right]^2
\]

Hence,

\[
\sum_{k=0}^{N-1} \left| I_{k,n} \right|^2 = \frac{1}{N^2} \left[ \frac{\sin \left( \frac{\pi (x_n f_s + \Delta f)}{N} \right)}{\sin \left( \frac{\pi (x_n f_s + \Delta f)}{N} \right)} \right]^2
\]

For small \(\Delta f/f_s\), the degradation due to the attenuation of the useful signal component and the interference from ICI become

\[
-10 \log_{10} \left( 1 - \frac{1}{3} (\pi_n f_s)^2 \right) \quad \text{and} \quad 10 \log_{10} \left( 1 + \frac{1}{3} E_s N_0 (\pi_n f_s)^2 \right)
\]

If the arguments of the logarithmic functions are close to 1, the degradations can further be expressed as

\[
\frac{10}{\ln 10} \frac{1}{3} (\pi_n f_s)^2 \quad \text{and} \quad \frac{10}{\ln 10} \frac{1}{3} E_s N_0 (\pi_n f_s)^2
\]

respectively.

It is obvious that when \(E_s/N_0\) is large, the degradation caused by ICI is dominant.