A Method of High-Precision Frequency Detection with FFT

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SUMMARY

Except for the nonlinear complex method such as MEM, there has not been known a method to determine the frequency with a high accuracy from the short wave train. Generally, it is considered that the frequency resolution in FFT is of the order of the step in the frequency. This paper attempts to derive a correction formula for FFT results by which the frequency, amplitude and phase of the arbitrary input signal with discrete frequencies can be calculated easily and with a high accuracy. The effect of the noise on the accuracy is discussed in the practical measurement condition, and it is shown by theory and numerical experiment for the case of white noise that the accuracy is linearly dependent on the noise ratio.

1. Introduction

The measurement technique by waves, such as sound and optical waves, is used widely. A signal processing is required which is common to those technologies to determine the frequency, phase and amplitude of the wave from the measured data. When a long wave train is available, the frequency can be determined with high accuracy by the zero-cross detection, and the phase can be determined by the phase detection.

However, it often happens that a sufficiently long wave train is not available. There have been no reports on the systematic method to determine the frequency, phase and amplitude of the periodic signal of a short wave train. This has been a problem in the design and evaluation of the method using the periodic signal of the short wave train.

It is well known that the frequency can accurately be determined by the maximum entropy method (MEM) from a signal of short wave train [1-4]. However, the method has some disadvantage in that the amplitude and phase cannot be determined, and the analysis becomes difficult due to its nonlinear nature, when a noise is included.

In the analysis with the fast-Fourier transform (FFT), the frequency resolution has been restricted to the inverse of the wave train length. Zero padding to the signal sequence to increase the apparent frequency resolution is used widely. However, the frequency resolution is improved only in proportion to the sequence length, which is a waste of memory space and the computational efficiency.

Other methods employing FFT have recently been reported by Hara et al. [7] and Parker et al. [8]. These methods are based on the interpolation of the amplitude near the maximum in the result of FFT to determine the frequency and amplitude of the periodic signal of the short wave train. Their methods are interesting in that the periodic signal of the short wave train can be analyzed easily and with a high accuracy. However, Hara's method is sensitive to the presence of noise and other frequency components since it does not use any window. Parker's method is effective in many cases, but the theoretical background is not clear. Furthermore, the determination of the phase is not considered in those reports.

This paper presents a strict correction formula which gives the accurate estimates for the frequency, the amplitude and the initial phase.

In the following, the correction formulas are derived, and the numerical experiments are presented for the case where the additive noise is included.
2. Derivation of FFT Response Function and Expression for Frequency

First, this section considers the determination of the frequency, amplitude and phase of the sinusoidal component from the input sequence without a noise. Section 4 analyzes the effect of the additive noise contained in the input sequence.

The complex sinusoidal input sequence with length $N$ can be written as follows:

$$x(i) = A e^{j2\pi f_0 i/N}$$

where $f$ is the frequency with the input sequence length as the unit; $A$ is the amplitude; and $f_0$ is the initial phase.

In the actual measuring system, the observed input sequence is usually a real sequence. However, one can write that

$$\cos(t) = 1/2 \left( e^{jt} + e^{-jt} \right)$$

By analyzing the input sequence as the complex, and by superposing the result for the positive and the negative frequencies with the same absolute values, the response to the real sequence can be calculated. Consequently, no generality is lost by analyzing the complex sequence. The case of the real input sequence is discussed in section 3.

To suppress the sidelobe which produces an undesirable effect in determining the frequency, a window must be operated on the input sequence. The Hamming window is used in this paper because of its simplicity in the processing [6, 8]. The Hamming window of length $N$ is represented as follows:

$$w(i) = 1 - \cos\left( \frac{2\pi i}{N} \right)$$

(This definition gives the double value of ordinary one, since we follow Parker [8].)

The input sequence $x(i)$ is multiplied with Hamming window $w(i)$. The discrete-Fourier transform $G(k)$ of the result is given as follows:

$$G(k) = \sum_{i=0}^{N-1} [w(i) x(i) e^{-j2\pi ki/N}]$$

Using the relation of Eq. (2), the Hamming window is summarized by a representation using exponents

$$G(k) = A e^{j2\pi f_0} \sum_{i=0}^{N-1} \left[ e^{j2\pi (f-k)} i/N - 1/2 e^{j2\pi (f-k+1)} i/N \right]$$

$$- 1/2 e^{j2\pi (f-k-1)} i/N$$

$$k=0, 1, \ldots, N-2, N-1$$

The geometrical series with ratio $a$ is calculated as follows:

$$\sum_{i=0}^{N-1} a^i = (a^n - 1)/(a-1) \quad (a \neq 1)$$

Furthermore, there exists the following relation:

$$e^{j\pi - 1} = (e^{j\pi 2} - e^{j\pi 2}) e^{j\pi 2}$$

Using those expressions,

$$G(k) = A e^{j2\pi f_0}$$

$$\times \frac{-\cos(\pi(f-k)/N) \sin^2(\pi/N)}{\sin(\pi(f-k-1)/N) \sin(\pi(f-k)/N)}$$

$$\times \frac{\sin(\pi(f-k+1)/N)}{\sin(\pi(f-k)) e^{j\pi(f-k)}}$$

$$k=0, 1, \ldots, N-2, N-1$$

This is the well-known response function of the discrete Fourier transform.

The response function for the input sequence is thus obtained.

When a similar analysis is made on the input sequence without multiplying the window, the following result $H(k)$ is obtained:

$$H(k) = A e^{j2\pi f_0}$$

$$\times \frac{-\cos(\pi(f-k)/N) \sin(\pi f/k)/N)}{\sin(\pi(f-k-1)/N) \sin(\pi(f-k)/N)}$$

$$\times \sin(\pi(f-k)) e^{j\pi(f-k)}$$

$$k=0, 1, \ldots, N-2, N-1$$

This is the well-known response function of the discrete Fourier transform.

Figure 1 shows the response functions obtained by Eqs. (8) and (9) for the case where $N$ is sufficiently large. In Eq. (9), the maximum of the sidelobe decreases, being inversely proportional to $|f-k|$. On the other hand, in Eq. (8), the maximum of the sidelobe decreases very rapidly, being inversely proportional to the third order of $|f-k|$, although the width of the main lobe is doubled. The sidelobe in the latter case can be regarded as almost 0 when $|f-k|$ exceeds 3.
Due to the forementioned property, the method using the window has the following two merits: the interpolation is easy near the maximum; and even if another frequency component is included in the input sequence, its effect does not affect greatly the response near \( f \), if the frequency is sufficiently apart.

The phase responses of Eqs. (8) and (9) differ by \( \pi(f-k)/N \) at the center of the mainlobe. This is due to the fact that the window (3) is used, with the value 0 at \( i = 0 \). To align the phase response, the window defined by

\[
w(i) = 1 - \cos\left(\frac{2\pi(i+1/2)}{N}\right)
\]

\((i=0,1,\ldots,N-2,N-1)\)

should be used. From the viewpoint of accuracy, the windows are equivalent. In the following, Eq. (3) is used in the analysis because of the simplicity of expression.

The suffix corresponding to the maximum absolute value in the sequence of \( G(k) \)
is written as $k_{\text{max}}$. Figure 2 shows the response of $G(k_{\text{max}})$; $f$ is contained in the following range:

$$k_{\text{max}} - 1/2 \leq f \leq k_{\text{max}} + 1/2 \quad (11)$$

If it is not the case, another $G(k)$ should have a larger absolute value, as is shown by the dashed line in the figure, which is a contradiction. When Eq. (11) is valid, $G(k_{\text{max}} - 1)$, $G(k_{\text{max}})$, and $G(k_{\text{max}} + 1)$ cannot be 0. By interpolating those values, $f$ can be determined.

Calculating the ratio $\gamma$ of $G(k_{\text{max}} - 1)$ and $G(k_{\text{max}})$ using Eq. (8),

$$\gamma = \frac{|G(k_{\text{max}} - 1)|}{|G(k_{\text{max}})|} = \frac{|\cos(\pi(f-k_{\text{max}}+1)/N)\sin(\pi(f-k_{\text{max}}-1)/N)|}{|\cos(\pi(f-k_{\text{max}})/N)\sin(\pi(f-k_{\text{max}}+2)/N)|} \quad (12)$$

When $N$ is sufficiently large, the following approximation can be made:

$$\cos(\theta) \approx 1 \quad (|\theta| < 1) \quad (13)$$
$$\sin(\theta) \approx \theta \quad (|\theta| < 1) \quad (14)$$

Substituting those expressions into Eq. (12), the interpolation formula is obtained:

$$\gamma \approx \frac{\pi(f-k_{\text{max}}-1)/N}{\pi(f-k_{\text{max}}+2)/N} = \frac{f-k_{\text{max}}-1}{f-k_{\text{max}}+2} \quad (15)$$

Determining $f$,

$$f = k_{\text{max}} + \frac{2\gamma-1}{\gamma+1} \quad (16)$$

When the approximations of Eqs. (13) and (14) are used, the error in $\gamma$ is less than $3 \times 10^{-5}$ for $N = 16$ and less than $2 \times 10^{-6}$ for $N = 32$. The error decreases with $N$ in proportion to $N^{-4}$, because of the Hamming window, and the expression is strict when continuous input is considered. For the reason described later, one can assume that $N \geq 32$. Considering that $\gamma$ is used in the interpolation between frequency steps, the approximation can be regarded as satisfactory.

Similarly, letting the ratio of $G(k_{\text{max}} + 1)$ and $G(k_{\text{max}})$ be $s$,

$$s = \frac{|G(k_{\text{max}} + 1)|}{|G(k_{\text{max}})|} \quad (17)$$

Equations (16) and (18) are valid simultaneously, and either can be used to calculate $f$. In the actual measurement, however, $G(k)$ contains a noise, and it is desirable to use the expression which is more insensitive to the noise. Differentiating Eq. (16) by $\gamma$,

$$df = \frac{-3}{(1+\gamma)^2} \quad (19)$$

It is seen from Eq. (19) that the absolute value of the derivative becomes smaller as $\gamma$ becomes larger, i.e., as $|G(k_{\text{max}} - 1)|$ becomes closer to $|G(k_{\text{max}})|$. A similar situation applies to $|G(k_{\text{max}} + 1)|$. Consequently, the effect of the noise in $G(k_{\text{max}} - 1)$, $G(k_{\text{max}})$, and $G(k_{\text{max}} + 1)$ can be minimized by
using Eq. (16), if \(|G(k_{\text{max}}+1)|\) is larger than \(|G(k_{\text{max}}-1)|\), and Eq. (18) otherwise.

To calculate the amplitude \(A\) and the initial phase \(P_0\), the approximations of Eqs. (13) and (14) are substituted into Eq. (8) to write the expression as

\[
A e^{j2\pi P_0} = - \frac{G(k_{\text{max}})}{N} \frac{\pi(f - k_{\text{max}})}{\sin(\pi(f - k_{\text{max}}))} \times (f - k_{\text{max}} - 1) (f - k_{\text{max}} + 1) e^{-j\pi(f - k_{\text{max}})}
\]

(20)

From the preceding expression, the amplitude and the phase are derived as

\[
A = - \frac{|G(k_{\text{max}})|}{N} \frac{\pi(f - k_{\text{max}})}{\sin(\pi(f - k_{\text{max}}))} \times (f - k_{\text{max}} - 1) (f - k_{\text{max}} + 1)
\]

(21)

\[
P_0 = \frac{1}{2\pi} \text{Arg}(G(k_{\text{max}}) e^{-j\pi(f - k_{\text{max}})})
\]

(22)

Thus, it is seen that \(f\), \(A\), and \(P_0\) can be calculated using Eqs. (12), (16), (17), (18), (21), and (22), from the sequence \(G(k)\) obtained by the discrete-Fourier transform using the Hamming window. Up to this point, it is assumed for the simplicity of the discussion that the input sequence has only one frequency component. As is seen from the response function of Fig. 1, however, two frequency components interfere little if their frequency is apart by 3 or more. Hence, even if the input sequence has a large number of discrete spectra, the frequencies of the components can be calculated separately, unless they are very close. It suffices to read the maximum in the foregoing discussion as extremum.

3. Effect of Real Input

Up to this point, the complex input sequence is assumed in the analysis for the simplicity of discussion. In most actual cases, however, a real sequence is obtained by measurement. In the following, the case of the input length of 16 is discussed.

The sinusoidal input sequence \(y(i)\) with frequency \(f\), amplitude \(A\), and initial phase \(P_0\) can be written as follows:

\[
y(i) = A \cos(2\pi(P_0 + if/N))
\]

\[
= \frac{A}{2} [e^{j2\pi(P_0 + if/N)} + e^{-j2\pi(P_0 + if/N)}]
\]

(23)

\((i=0,1,\ldots,N-1,N-2)\)

As is seen from this expression, the real input sequence \(y(i)\) can be written as superposition of the complex input sequence with amplitude \(A/2\) and frequency \(f\), and the complex input sequence with amplitude \(A/2\) and frequency \(-f\). Because of this situation, \(G(k)\) response not only to the positive frequency component, but also to the negative frequency component. In the mid-frequency, the response to the negative frequency component is sufficiently small compared with that to the positive one, and can be ignored.

In the extremely low-frequency (\(f < 2\)), however, or the frequency close to Nyquist frequency (\(f > N/2 - 2\)), the responses to the positive and the negative frequencies interfere on the mainlobe, making it difficult to calculate the accurate value by the proposed method. Figures 3(a) and (b) show this situation, for the case of \(k = 1\) and \(k = N/2 - 1\), respectively. As is seen from the figure, there exists a symmetry in the phenomenon at both ends. An investigation was made for the minimum-required length of the input sequence, and it is seen, for the case of the input length of 16, that the frequency is 0 to 8, and 1 to 2 steps at both ends cannot be utilized. This indicates a certain limit concerning this point.

4. Effect of Noise on Accuracy

The actually observed data contain noise. In the following, the effect of the noise contained in the input sequence on the accuracy is discussed. Considering the actual situation, the analysis is made for the real input sequence. It is assumed that the noise is white and additive.

The real input sequence \(s(i)\) containing noise is represented as follows:

\[
s(i) = x(i) + n(i)
\]

\((i=0,1,\ldots,N-2,N-1)\)

(24)

where \(x(i)\) is the sequence representing the signal without noise. It is assumed that \(n(i)\) is the noise component with standard deviation \(\sigma_n\) and mean 0, which is independent if \(i\) is different.

Assuming that \(x(i)\) is a sinusoidal wave with amplitude \(A\), SN ratio \(R\) is given by

\[
R = \frac{A / \sigma_n}{\text{SNR}}
\]

(25)

Based on the foregoing expression, the representation for the noise spectrum is derived. Then it is determined how \(R\) affects the error in the frequency, amplitude and phase.

The discrete-Fourier transform \(G(k)\) of Eq. (25) is given by
The variance of $Ar$ is given by

$$G(k) = G(k) + N(k)$$

$(k = 0, 1, \ldots, N-2, N-1)$

$G(k)$ is given by Eq. (8). As in Eq. (4), $N(k)$ is given by

$$N(k) = \frac{1}{N} \sum_{i=0}^{N-1} w(i) n(i) e^{-j2\pi ik/N}$$

$(k = 0, 1, \ldots, N-2, N-1)$

Since $\overline{\langle n^2 \rangle} = 0$ (where $\overline{\langle \rangle}$ indicates the expectation),

$$N(k) = 0$$

and the variance $\overline{\langle N(k)^2 \rangle}$ is given as follows:

$$\overline{\langle N(k)^2 \rangle} = \left| \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} w(i) w^*(j) n(i) n^*(j) \right| e^{-j2\pi (i-j)/N}$$

$(k = 0, 1, \ldots, N-2, N-1)$

(The asterisk indicates the complex conjugate, and does not affect the value for the case of a real sequence.)

Since

$$\overline{\langle n^2 \rangle} = \sigma^2$$

$(i = i)$

and

$$\overline{\langle n^2 \rangle} = 0$$

$(i \neq i)$

It follows that

$$\overline{\langle N(k)^2 \rangle} = \sigma^2 \sum_{i=0}^{N-1} \left| w(i) \right|^2$$

$$= \sigma^2 \sum_{i=0}^{N-1} (1 - \cos(2\pi i/N))^2$$

$$= \sigma^2 \frac{3}{2} N$$

Using the foregoing expression, the error in $\alpha$ in Eq. (12) is examined. Let the error in $\alpha$ be $\Delta \alpha$. Since $\overline{\langle N(k)^2 \rangle} \ll \overline{\langle G(k)^2 \rangle}$

$$\Delta \alpha = \frac{\overline{\langle G(k_{max} - 1) + N(k_{max} - 1) \rangle}}{\overline{\langle G(k_{max}) + N(k_{max}) \rangle}}$$

$$= \frac{\overline{\langle G(k_{max} - 1) \rangle}}{\overline{\langle G(k_{max}) \rangle}} \frac{\overline{\langle 1 + N(k_{max} - 1)/G(k_{max}) \rangle}}{\overline{\langle 1 + N(k_{max})/G(k_{max}) \rangle}}$$

$$\approx r \left[ 1 + \frac{N(k_{max} - 1)/G(k_{max})}{N(k_{max})/G(k_{max})} \right]$$

$$\approx r \ Re \left[ \frac{N(k_{max} - 1)}{G(k_{max} - 1)} - \frac{N(k_{max})}{G(k_{max})} \right]$$

($\Re$ indicates the real part. Consequently, $\alpha$ is given by

$$\Delta \alpha = r \ Re \left[ \frac{N(k_{max} - 1)}{G(k_{max} - 1)} - \frac{N(k_{max})}{G(k_{max})} \right]$$

and the indirect effect in the calculation of the frequency. For simplicity, the error of the amplitude is evaluated as a simple sum of the two. In Eq. (21), the maximum absolute value of the derivative of the term through which the frequency $\alpha$ affects the amplitude error is $\pi/4$. Consequently,

$$\frac{\sigma_{\alpha}}{A} = \left[ \left\{ \Re \left( \frac{N(k_{max})}{G(k_{max})} \right) \right\}^2 + \pi \sigma_{\alpha} \right]^{1/2} + \pi/4$$

$$= \frac{2.6}{N^{1/2}}$$

The standard deviation $\sigma_{\alpha}$ for the calculated frequency $\alpha$ is given by

$$\sigma_{\alpha} = \frac{2.6}{N^{1/2}}$$

where $\Re$ indicates the real part. Consequently, $\alpha$ is given by

$$\Delta \alpha = r \ Re \left[ \frac{N(k_{max} - 1)}{G(k_{max} - 1)} - \frac{N(k_{max})}{G(k_{max})} \right]$$

and the indirect effect in the calculation of the frequency. For simplicity, the error of the amplitude is evaluated as a simple sum of the two. In Eq. (21), the maximum absolute value of the derivative of the term through which the frequency $\alpha$ affects the amplitude error is $\pi/4$. Consequently,

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$$= \frac{2.6}{N^{1/2}}$$

The variance of $\Delta \alpha$ is given by

$$\overline{\langle \Delta \alpha^2 \rangle} = \overline{\langle \Delta \alpha^2 \rangle} = \overline{\langle \alpha^2 \rangle} / \overline{\langle \alpha^2 \rangle}$$

In the derivation of Eq. (30) from Eq. (25), it is assumed that the phase of the noise does not have any tendency, and the variance of the real part is set as one-half the variance of the original complex number. In the derivation of Eq. (30) from Eq. (25), the ratio of the two spectra is assumed as independent, which is not correct from a strict viewpoint. Since a window is provided to the input sequence, the ratio of two spectra exhibits a positive correlation. Because of this situation, Eq. (30) gives a slight overestimation. However, since the aim of this paper is to set the upper limit for the error, no further complicated analysis is made.
Similarly, for the phase,
\begin{align*}
\sigma_{\phi} &= \frac{1}{2\pi} \left[ \left( \frac{\sqrt{\text{Im}(N G_{\max})}}{G_{\max}} \right)^2 \right]^{0.5} + \frac{\pi}{2\pi} \sigma_f \\
&= \frac{1}{N^{0.5}}
\end{align*}

(40)

It is seen from Eqs. (38) to (40) that the accuracies in frequency, amplitude and phase are inversely proportional to the SN ratio, and are inversely proportional to the square root of the number of data \( N \). The frequency in this paper implies the frequency in regard to the input frequency length. Consequently, for the input sequence with the same sampling frequency, \( f \) increases in proportion to the \( N \) of data. Because of this property, the relative accuracy of \( f \) is improved inversely proportional to 3/2 power of \( N \).

Letting, for example, \( f = N/4 \),
\[
\frac{\sigma_f}{f} = \frac{1.8}{(N/4)^{0.5} R} = \frac{7.2}{N^{1.5} R}
\]

(41)

In other words, for the input sequence with given time length, if the data length \( N \) is increased while keeping the total observation length constant, the accuracy is improved only in proportion to \( N^{-1/2} \), which is the statistical effect. When the sampling rate is fixed and the number of data is increased (the observation interval increases in proportion to \( N \)), the accuracy is improved in proportion to \( N^{-3/2} \).

Consequently, in the measurement of a continuous signal, it is advantageous to observe the wave for a long time, making the sampling rate slower, as far as the Nyquist rate permits. However, it should be noted that as is seen from the discussion in the previous section, the error increases as one approaches the Nyquist rate.

5. Example of Numerical Experiment

To verify the validity of the theory, a numerical simulation was performed. First, the determination of the frequency, amplitude and phase for the case without noise was examined. The input sequence is composed of 64 points, and the frequency was varied from 0 to 32, corresponding to the Nyquist rate. The reason for setting the number of data as small as 64 is that the behavior near the ends should be shown clearly on the figure.

Figure 4 shows the result. The initial phase is set as 0, and the frequency is varied with the step of 0.1. The errors in the frequency, amplitude and phase are shown. As is seen from the discussion in section 3, the error increases rapidly for \( f < 2, f > 30 \), where the main lobe of the response function responds also to the negative frequency. However, little error is observed for \( f < 5, f > 27 \). It is thus seen that the measurement can be made for the range \( 2 \leq f \leq N/2-2 \), when a high accuracy is not required. If a high accuracy is required, it suffices that \( f \) is 5 to 6 steps apart from the end. It is seen that the curves have a symmetry near 0 and \( N/2 \).

Next, the case with noise is calculated to examine the effect of noise on the measurement accuracy. For three cases of data lengths 64, 256 and 1024, the frequency is given at random in the range \( 6 < f < N/2 - 6 \), and the initial phase \( \phi_0 \) is also given at random in the range \( -1/2 \leq \phi_0 < 1/2 \). One hundred trials were made for each case, and
The mean of standard deviation is shown in Fig. 5. The white noise is simulated by the uniform pseudorandom number with mean 0.

It is seen that the curves for the frequency, amplitude and phase have the slope of -1. Since the accuracy for three calculated cases differs by a factor of 2, it is seen that the accuracy is proportional to $N^{-1/2}R$, as is expected from the analysis of section 4. Some fluctuations in the curves are due to the relatively small number of trials, i.e., 100. The proportional constant is also determined by 1000 trials for SN ratio of 10 dB. The results were 1 for frequency, 1.3 for amplitude, and 0.6 for phase, which are approximately one-half the result of analysis in section 4. This is due to the excessive estimation for the error in the analysis. Especially, for the amplitude and phase, the error was estimated as a simple sum of the errors of $G(k_{max})$ and the calculation error for the frequency, leading to a larger value. The analysis in section 4 can, however, be considered as an upper bound for the error.

The accuracy in the actual situation is estimated as follows. For the case where the number of data is 1024 and the frequency is approximately 300, it is seen from the curve that the accuracy of the order of 1.5 below the decimal can be achieved for the frequency. This is more than four orders of magnitude as the relative accuracy for the frequency. If SN = 20 dB, the accuracy of five orders of magnitude is secured. The accuracy for the amplitude and phase are of the same order. In the actual situation, a larger SN ratio can easily be obtained. Considering that the frequency is 300 during the measurement which is of only three digits, it is seen that the proposed method has a very high accuracy compared with other methods, such as zero cross.

6. Conclusions

The resolution in the frequency measurement by FFT has been considered as of the same order as the step of the frequency. However, it is shown in this paper that the relative accuracy can be improved up to the rounding error (7 to 8 orders of magnitude) by correction. It is verified that the method is stable, and five or more orders of magnitude are obtained when an oscillator is connected directly to an A/D converter with 256 point input, and four to five orders of magnitude are obtained when the ultrasound in air is actually measured. The special features of the method are that the frequency, amplitude and phase can be obtained with sufficient accuracy from a short-wave train (several periods are sufficient). In addition, the phase information can be obtained from the measurement of the burst, which is useful in comparing the theory and experiment in the sound propagation.

As is well known, the computational complexity of FFT is proportional to $N\log(N)$. When a single frequency is to be measured, the computational complexity is increased by the factor of $\log(N)$. However, the method is useful considering that a large number of
frequency components can be determined simultaneously with a high accuracy. Furthermore, FFT is implemented on a hardware, and the method can be obtained with the real time processing or measuring equipment.

REFERENCES


AUTHORS (from left to right)
