G. Blahó and I. Kollár, "Frequency Limit Extension of Digital Fourier Analyzers." 9th IMEKO World Congress, West-Berlin, May 24-28. 1982. ACTA IMEKO 1982, North Holland, Amsterdam, pp. 187-196.

FREQUENCY LIMIT EXTENSION OF DIGITAL FOURIER ANALYZERS

by

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In some correlators vernier sampling /two samplers having slightly different sampling intervals/ is used to increase time resolution. The paper suggests a direct method with about the efficiency of the FFT for processing the vernier sampled data to obtain the power density spectrum. By this the bandwidth of a microprocessor-controlled spectrum analyzer can be increased to its 10 to 50 fold. The paper derives the formulae of the computing algorithm, analyses the time and spectral window functions, and deals with the variance and distortion of the spectral estimator. The algorithm requires a somewhat more complex program and a controller capable of vernier sampling - neither of these increases greatly the price of the analyzer.

Keywords: non-uniform sampling, Fourier analyzer digital spectrum analyzer

1. INTRODUCTION

Nowadays Fourier analysis at audio or lower frequencies is generally performed by means of digital analyzers. Because of the efficiency of the FFT algorithm /3/ digital analyzers are often based on FFT processors. However, in Fourier analyzers several-bit A/D converters are used in order to decrease variance and distortion, and the A/D converters of moderate price set the limit of the bandwidth to 25-4o kHz in the case of uniform sampling. Since random sampling methods are generally less effective as compared with FFT, only special deterministic sampling strategies, e.g. vernier sampling /Fig.1/ give the possibility of increasing the bandwidth without using high-speed A/D converters. An efficient algorithm for directly computing power density spectrum from vernier sampled data can greatly increase the performance of the "vernier" analyzers.

2. DEVELOPING THE ALGORITHM

Let us consider the two series of data in Fig.2. The signals x/t/ and y/t/ are sampled in /p+1/q and pq points respectively. The A/D conversion time can not be greater than the time resolution:

$$t_{AD} \leq p\Delta t.$$

In the case of uniform sampling this would limit frequency bandwidth to

$$\frac{1}{2t_{AD}}.$$

Considering the case of Fig.2 it can be easily stated that the correlation function

$$R_{xy}(\tau) = E\{x(t)y(t+\tau)\}$$

can be estimated from the two series in the $k\cdot\Delta t$ points /k=0,1,-1,2.../, that is, the time resolution (Δt) allows a frequency bandwidth of

$$\frac{1}{2\Delta t}.$$

Let us remember now the direct Fourier Transform /DRFT/ method:

$$\hat{S}(k\Delta f) = \frac{\Delta t}{N}\overline{X(k\Delta f)}Y(k\Delta f),$$

and try to find a similar formula. Let us introduce the following notations:

$$R(k) \triangleq R(k\Delta t) \qquad S(k) \triangleq S(k\Delta f)$$

$$\Delta f = \frac{1}{T} = \frac{1}{Q\Delta t} \qquad x_i \triangleq x(i\Delta t)$$

$$Q = qp(p+1) \quad M = q(p+1) \quad N = qp$$

$$X_L(k) = \sum_{i=0}^{L-1} x_i e^{-j2\pi\frac{ki}{L}} \quad \text{/L is the sequence length/}$$

We prove in the following that

$$\hat{S}(k) = \frac{\Delta t}{q}\overline{Z_M(k)}V_N(k) \quad \text{/k=o...Q-1/} \qquad /1/$$

is an estimator for $S(k)$.

$$E\left\{\hat{S}(k)\right\} = \frac{\Delta t}{q} E\left\{\sum_{m=0}^{M-1} z_m e^{j2\pi\frac{km}{M}} \sum_{n=0}^{N-1} v_n e^{-j2\pi\frac{kn}{N}}\right\} =$$

$$= \frac{\Delta t}{q} E\left\{\sum_{m=0}^{M-1} x_{mp} e^{j2\pi\frac{kmp}{Q}} \sum_{n=0}^{N-1} y_{n(p+1)} e^{-j2\pi\frac{kn(p+1)}{Q}}\right\} =$$

$$= \frac{\Delta t}{q} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} E\left\{x_{mp} y_{n(p+1)}\right\} e^{-j2\pi\frac{k[n(p+1)-mp]}{Q}} =$$

$$= \frac{1}{q} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} R_{xy}\left(n[p+1]-mp\right) e^{-j2\pi\frac{k[n(p+1)-mp]}{Q}} \Delta t. \quad /2/$$

Eq.2 clearly shows that the expected value of $\hat{S}(k)$ is a weighted Fourier transform of the correlation function. The weighting function /time window, $w(\tau)$ / is normalized: from Fig.2 it can be seen that the argument $(n[p+1]-mp)$ is exactly in q cases equal to zero, so dividing by q $w(0)=1$. $w(\tau)$ is treated in detail in Sect.4.

Summing up the above statements, the algorithm consists of the following main steps:

- computation of the $\{Z_k\}$ and $\{V_k\}$ discrete Fourier transformed series /$0 \leq k < M$ and $0 \leq k < N$ respectively/;
- computation of the power density spectrum estimator

$$\hat{S}(k\Delta f) = \frac{\Delta t}{q} \bar{Z}_k V_k$$

  for the values $0 \leq k < Q$. For the values $k > M$ or $k \geq N$ we make use of the fact that

$$Z_k \equiv Z_{k(\mathrm{mod}M)} \quad \text{and} \quad V_k \equiv V_{k(\mathrm{mod}N)} .$$

Here $k(\mathrm{mod}M)$ denotes the remainder when dividing k by M.

- Repeating the foregoing steps and forming a mean value in order to decrease the variance.

3. SPEED OF THE ALGORITHM

The run time of an algorithm is often supposed to be proportional to the number of multiplications. In this section the suggested algorithm is compared with the algorithm used in indirect spectrum analysis by comparing the numbers of multiplications. Although both M and N cannot be the power of 2 at the same time, an FFT can be executed on prime number base /4/, or the Winograd Fourier transform /5,6/ can be used. So one can expect that the speed increase is about the same as between direct and indirect Fourier analysis.

Examples of comparison /values of p,q,M,N and Q/ are listed in Table 1. These numbers are chosen in such a way that the prime factors of M,N,Q are possibly small.

p or frequency limit multiplier	q	M=q(p+1)	N=qp	Q=qp(p+1)
$2^3=8$	$2^4=16$	$2^4 \cdot 3^2=144$	$2^7=128$	$2^7 \cdot 3^2=1152$
$3 \cdot 5=15$	$2^3=8$	$2^7=128$	$2^3 \cdot 3 \cdot 5=120$	$2^7 \cdot 3 \cdot 5=1920$

Table 1 Examples of constants

For simplicity, let the number of complex multiplications be approximated by $N\log_2 N$ in an FFT algorithm of N elements even if N is not a power of 2. One complex multiplication is equivalent to four real multiplications.

a./ To compute the correlation function estimator, the number of real multiplications required is MN. The transform executes $Q\log_2 Q$ complex multiplications.

b./ In the suggested algorithm the transforms require $M\log_2 M + N\log_2 N$ complex multiplications, and further Q complex multiplications are needed to compute the power density spectrum estimator.

Table 2 lists the numbers of real multiplications for the two algorithms and the two examples. The suggested algorithm is of similar efficiency as the Cooley-Tukey algorithm, corresponding to expectations.

M	N	Q	Indirect alg.	Suggested alg.
144	128	1152	$6,5\cdot 10^4$	$1,2\cdot 10^4$
128	120	1920	$8,4\cdot 10^4$	$1,4\cdot 10^4$

Table 2 The number of real multiplications

4. DISTORTION

In order to obtain an expression for the distortion, the window function should be investigated. Considering Eq.2 and Fig.2 it can be stated that the value of $q\cdot w(k\Delta t)$ is equal to the number of occurrences of the $[x(s\Delta t),\ y(s\Delta t+k\Delta t)]$ pairs. The shape of $q\cdot w(k\Delta t)$ can visually be obtained from Fig.2 /Fig.3/. In $q\cdot w(k\Delta t)$ a step function dominates /Fig.3b/, and an additional ripple function /with values of 0 and 1/ can be observed. For the values of $w(k\Delta t)$ the following inequalities hold:

$$q-\left[\frac{k}{p(p+1)}\right]-1 \le q\cdot w(k\Delta t) \le q-\left[\frac{k}{p(p+1)}\right] \qquad /0\le k<Q/ \qquad /3/$$

where $[x]$ denotes the integer part of x.

The step function $/w_s/$ can be expressed as:

$$T_w = p(p+1)\Delta t = \frac{T}{q}$$

$$w_{T_w}(t) = \varepsilon(t+T_w)-\varepsilon(t-T_w)$$

$$\varepsilon(t)=\begin{cases} 0 \text{ if } t<0; \\ 1 \text{ if } t\ge 0. \end{cases}$$

$$w_s(t) = \sum_{i=1}^{q-1} \frac{1}{q}\, w_{i\cdot T_w}(t). \qquad /4/$$

The values of the window function can be computed analytically as follows. The value of $q\cdot w(k\Delta t)$ equals the number of the corresponding x,y pairs, that is, the number of solutions of the following Diophantine equation /2/:

$$k = n(p+1)-mp\ ,\quad 0\le m<q(p+1),\quad 0\le n<qp. \qquad /5/$$

To solve the equation, it has to be transformed:

$$m = \frac{n(p+1) - k}{p} = n + \frac{n-k}{p} \quad . \qquad /6/$$

Here $\frac{n-k}{p}$ must be integer. Consequently:

$$n = k(\mathrm{mod}\,p) + rp \ , \qquad 0 \le r < q \ . \qquad /7/$$

Substituting Eq.7 into Eq.6:

$$m = k(\mathrm{mod}\,p) + rp + r - \left[\frac{k}{p}\right] = r(p+1) + t \qquad /8/$$

where $t = k(\mathrm{mod}\,p) - \left[\frac{k}{p}\right]$.

For a given k and p exactly q values of m can be computed /see the inequality for r in Eq.7/. Solutions are those values of m, for which the inequality in Eq.5 holds. The number of solutions is obviously equal to the length of the intersecting part of the two sections in Fig.4.

Considering the distortion, in most cases the ripple function can be neglected and the window function can be approximated by the Bartlett window. So the distortion is about the same as by the DRFT:

$$E\{\hat{S}(f)\} = W(f) * S(f) \approx W_B(f) * S(f) .$$

The Fourier transform of the window and of the step function with the parameters given in the example are plotted in Fig.5.

5. VARIANCE

It is shown in Section 2 and Section 4 that the algorithm is in close relationship with the direct Fourier Transform /DRFT/ method. For stochastic signals DRFT has 100% variance /1/. Considering that we form the product of two Fourier transformed series as well, the variance is also approximately 100% in our case.

For deterministic signals /e.g. sine wave/ DRFT has rather small variance /caused by the unknown phase/. Investigating our algorithm we have found that this case has some special variance problems. An example is shown in Fig.6.

A sine wave of 3Hz is measured with parameters t=0,05s , p=4, q=3. The sampling rates are correspondently 4Hz and 5Hz. On Fig.6 $\{Z_k\}$ and $\{V_k\}$ are drawn; the repeated spectra caused by small sampling rates can be observed. It can be noticed that the peaks coincide not only at 3Hz but at 7Hz as well. Since the phase of the product at 7Hz rotates when the phase of the sine wave is shifted, the mean is zero as expected, but the variance is great: the relative variance is in this case infinite. This means that the "vernier spectra" should be averaged to avoid false peaks. On the other hand, this is not a very awkward requirement: when measuring spectra at higher frequencies the required record length for one estimate is relatively short.

The situation shown for deterministic signals can naturally occur in the case o stochastic signals as well. So the variance at certain frequencies can be increased because of other spectrum peaks.

7. SUMMARY

The theoretical basis for directly computing the power density spectrum from vernier sampled data is explained. The suggested algorithm has a similar efficiency as FFT. The window function, distortion and variance are treated in detail. Further work is needed to thoroughly investigate the variance and the points of view for choosing p and q, and to prove the advantages of the algorithm in practice.

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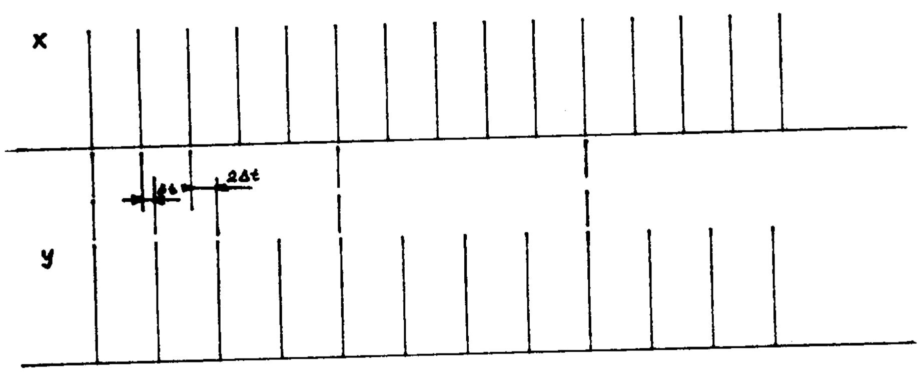

Figure 1. Timing diagram of vernier sampling
/p=4, q=3/

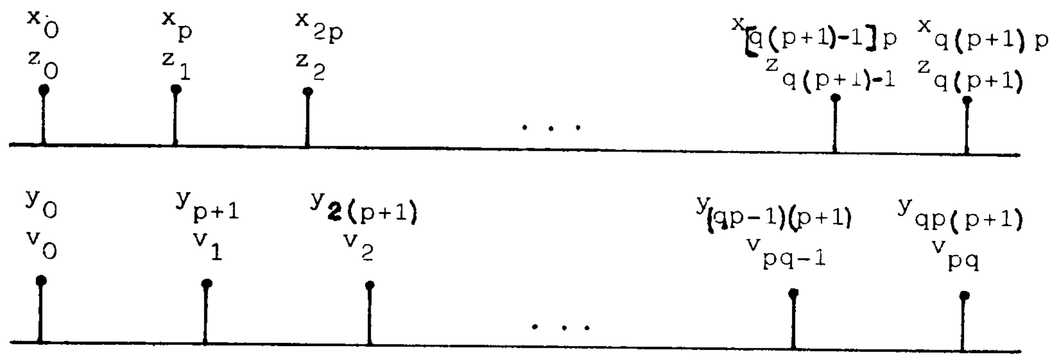

Figure 2. Sampled series

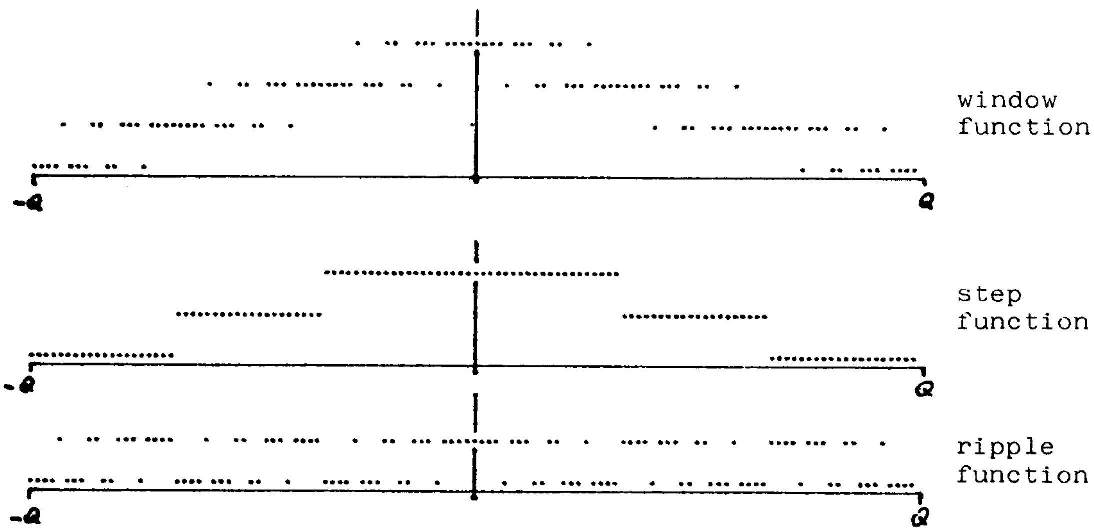

Figure 3. The window function /p=4, q=3/

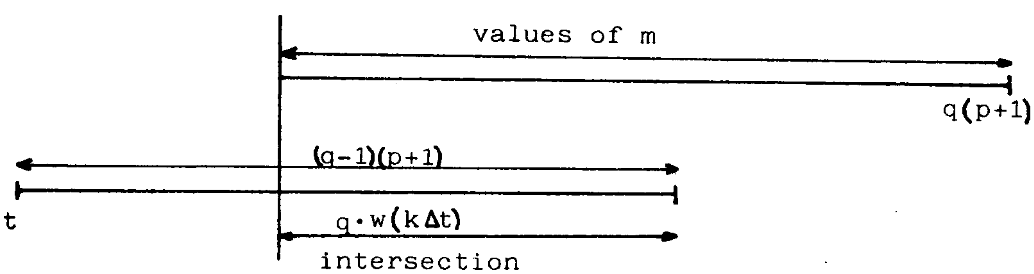

Figure 4. Solution of the Diophantine equation

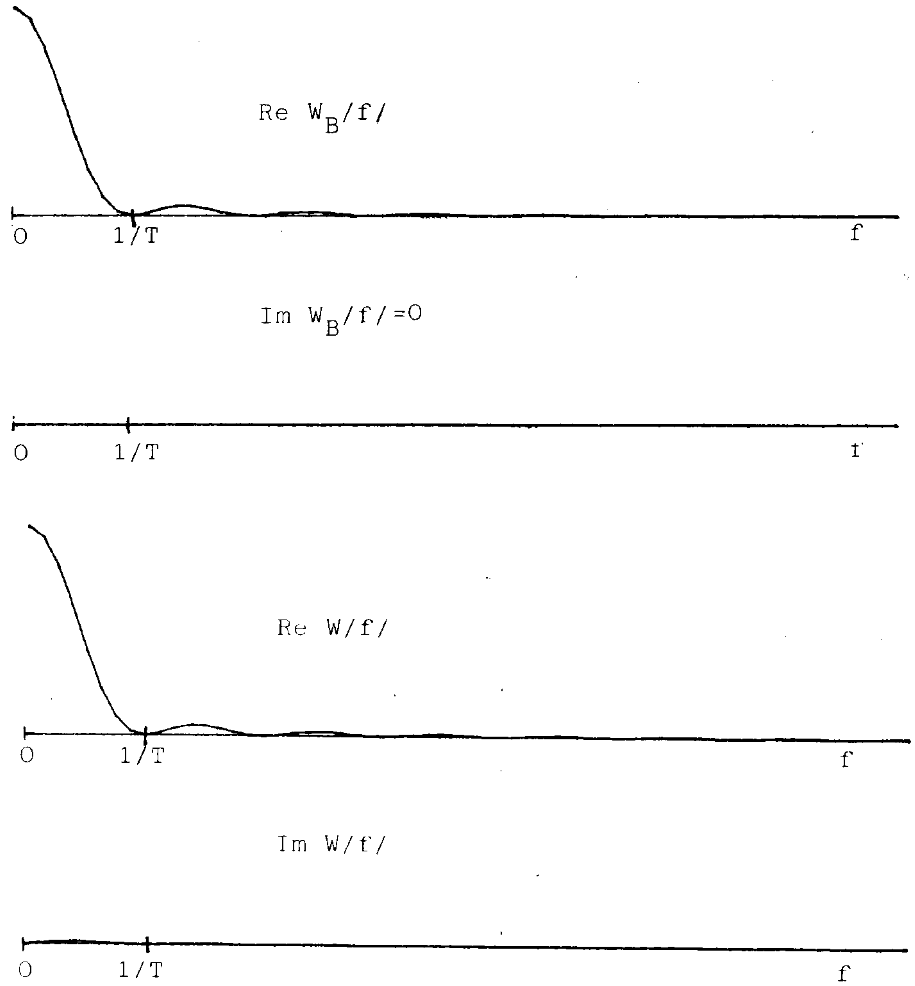

Figure 5. Windows in frequency domain

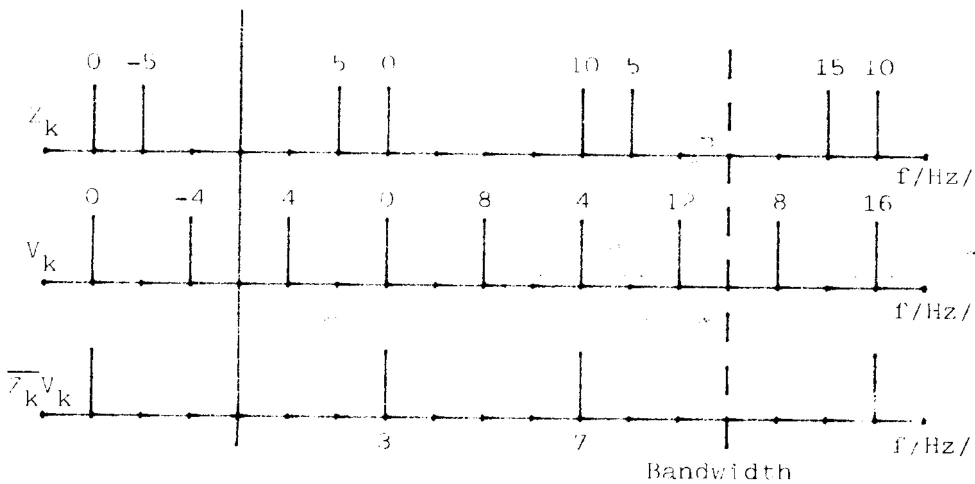

Figure 6. Variance problem
/Δt=0.05s, p=4, signal: 3Hz sine wave/