STEADY-STATE VALUE MEASUREMENTS WITH QUANTIZED DATA*

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1 Introduction

In several measurements like temperature or weight measurements the point of interest is only the steady-state value of the transient response. The most often used method is to wait for the steady-state conditions and then to measure the constant signal with the desired accuracy. However, this procedure has some disadvantages:

- The measurement time depends on the duration of the transients, so it may be rather long;
- Though the transient period of the signal contains information concerning the steady-state value as well, we do not make use of it, and so the variance due to measurement noise is not eliminated really effectively.

A method to overcome the above problems may be as follows. The transient response of a linear system can generally be modeled as the sum of the steady-state value, some complex exponentials and some noise. E.g. in the case of a second-order oscillating system:

$$x(t) = x_\infty + Ae^{-t/T}\cos\omega_N t + Be^{-t/T}\sin\omega_N t + n(t)\,. \tag{1}$$

Here A and B are parameters which describe the generally unknown initial conditions of the system. Having sampled a certain period of the transient response, model (1) can be fitted to the measured series, and the unknown parameters may be determined. If the coefficients of the differential equation of the system are known (this means that ω_N and T can be computed), and no assumptions for $n(t)$ are made, the least-squares method may be used for the estimation of x_∞, A and B. A non-recursive form for the estimation of the steady-state value is

$$\hat{x}_\infty = \sum_{i=1}^{N} a_i x_i\,, \tag{2}$$

where x_i denotes the i^{th} sampled value, and the parameters a_i are calculated by using the LS method [1,2].

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Formula (2) shows a weighted averaging of the observed series. This gives the idea that (2) could increase the resolution of $\hat{x}_\infty$ even in the case of coarsely quantized $\{x_i\}$ series. In the following we shall deal with this question in detail.

2 Modeling the Quantization Error

If the quantum size is small enough compared to signal variations, quantization may be well modeled by means of an *additive, uniformly distributed noise* ([3]; *Fig. 1*). This can be well observed in *Fig. 3*: the major part of the quantization error ($n_q(t)$ is very similar to a sawtooth signal which has uniform distribution. Deviations occur in the regions where the signal variation is small. When the transients are over, the uniform distribution model is not valid any more, since there is no variation (disregarding the noise).

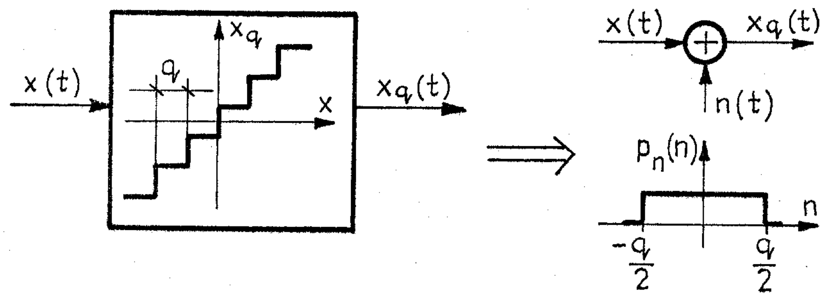

Figure 1: The noise model of quantization

Let us investigate now the sampling of the quantization error. If the sampling frequency is not too high in comparison to the 'sawtooth-frequency', the noise samples are approximately independent, that is, the sampled noise is approximately *white*. However, in the periods of small signal variations the assumption of independence is not valid, that is, the white noise model may be used *in the transient period* and *for limited sampling frequency* only.

If the uniform distribution white noise model may be used, the quantization-caused variance of $\hat{x}_\infty$ can be expressed as follows:

$$\text{var}\,\{\hat{x}_\infty\} = \left(\sum a_i^2\right)\frac{q^2}{12} \approx \frac{\text{const}}{N}\frac{q^2}{12}\,. \tag{3}$$

That is, for the signal $x(t)$ of *Fig. 3* and for the given observation time T_m we expect that increasing N the variance decreases due to the averaging. However, for large N's the model is not valid any more. For large sampling frequencies the samples describe the quantized signal very well, so the change of N does not influence the variance any more, and a diagram like *Fig. 2* may be expected (continuous line). This 'saturation' can be eliminated by using dither [3]. If before quantization we add a white noise $n_d(t)$ with uniform distribution between $[-q/2, q/2]$ to the signal, it 'randomizes' the quantization noise, and so the uniform distribution white noise model is valid for any N. In this case the variance is (see *Fig. 2*, dashed line):

$$\text{var}\,\{\hat{x}_\infty\} = \text{var}\,\{n_d(t)\} + \text{var}\,\{n_q(t)\} = 2\left(\sum a_i^2\right)\frac{q^2}{12}\,.$$

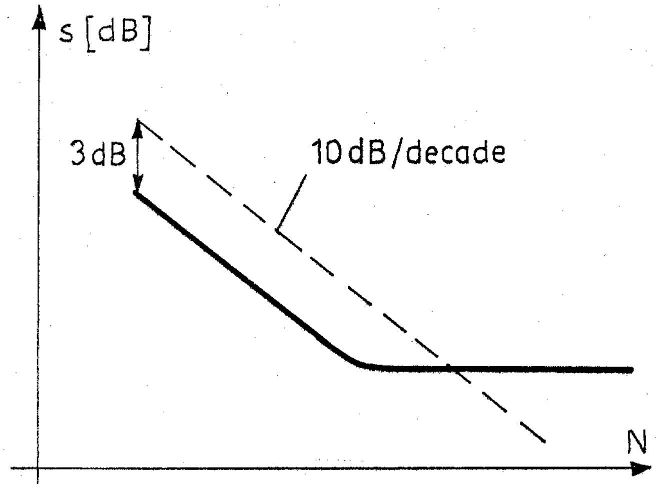

Figure 2: The expected behavior of the standard deviation

3 Simulation Results

The above considerations have been checked by simulation for the signal of *Fig. 3* with the following parameters:

$\omega_N = 2\pi f_0 \sqrt{1-\xi^2}$, $f_0 = 10$ Hz , $\xi = 0.2$, $T = 0.083$ s ,

$T_m = 0.15$ s (measurement time, observation time),

$x_\infty = 1$ (steady-state value),

$n(t) = n_q(t)$ or $n(t) = n_d(t) + n_q(t)$ (see above).

The initial conditions were slightly random:

$$E\{x(0)\} = 0 \ , \quad E\{x'(0)\} = 0 \ , \quad \sigma_A = \sigma_B = 0.1 \ , \quad A, B \text{ Gaussian} \, .$$

The stochastic nature of the initial conditions does not influence much the variance, since the use of (2) eliminates its effect by the use of the LS fitting. Still this randomness makes possible to measure the variance due to quantization.

First of all the series $\{a_i\}$ have been determined (see *Fig. 3*). In the simulated case the values a_i/N were nearly the same in corresponding time points for every N, so for every N, the value $N \sum a_i^2$ was almost the same, approximately 1.13.

Simulation has been proceeded 200 times for every $N-q$ pairs. In *Fig. 4* the confidence intervals for the empirical standard deviation are plotted. Confidence level: 95%, $\hat{x}_\infty$ was assumed to be Gaussian. Theoretical results are drawn with continuous line. The dotted line shows the saturation effect described in Sect. 2. (see *Fig. 2*). The measured distortion was in every case smaller than the empirical standard deviation.

It is surprising that the white noise model still seems to be valid even for large values of N. In the case of the simulated signal (*Fig. 3*) for 4-bit quantizer ($M = 16$) the limit is about $N \sim 150$ (see *Fig. 4*). This means that algorithm (2) is a very good means to increase the resolution. The explanation may be the following: the transient itself *acts as dither* from the point of view of the measurement of the steady-state value: the effect of the dominant wave is eliminated by means of the (2) algorithm, and its quantization error acts as uniform distribution dither.

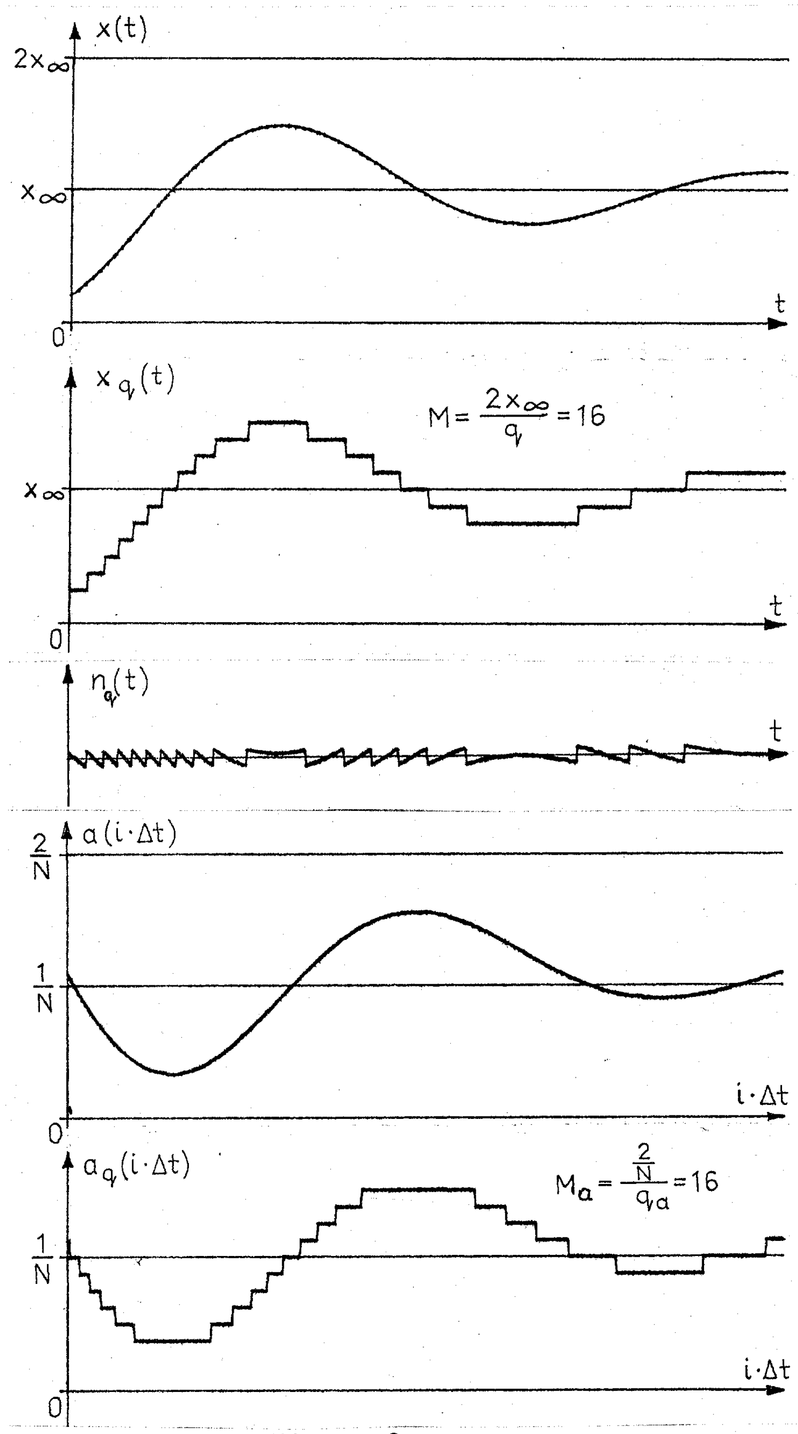

Figure 3: The transient response, its quantization and the weighting function

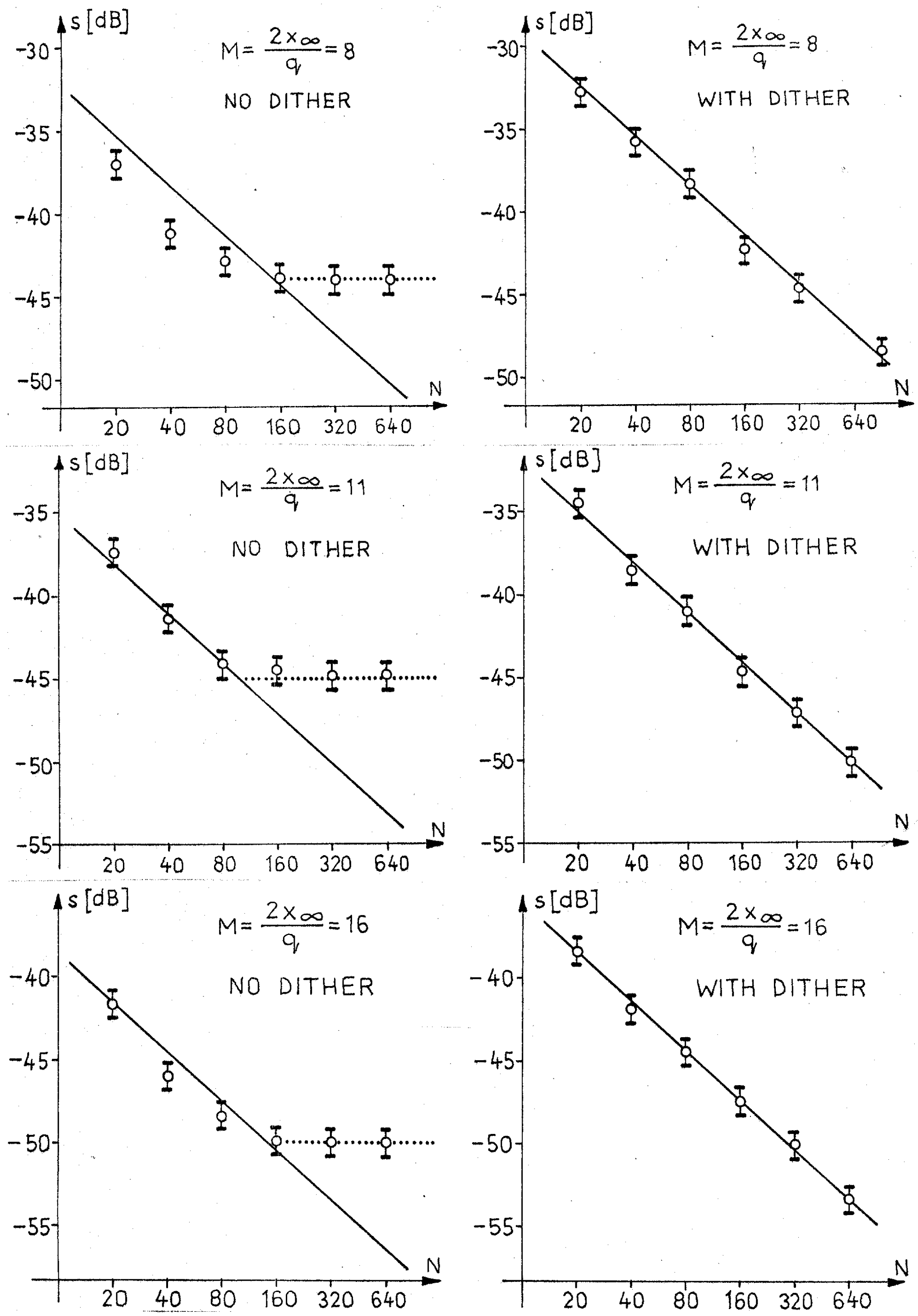

Figure 4: Empirical standard deviations for $M = 8, 11, 16$

In *Fig. 5* we have plotted the beginning point of the saturation effect as a function of

$$M = \frac{2x_\infty}{q} \, .$$

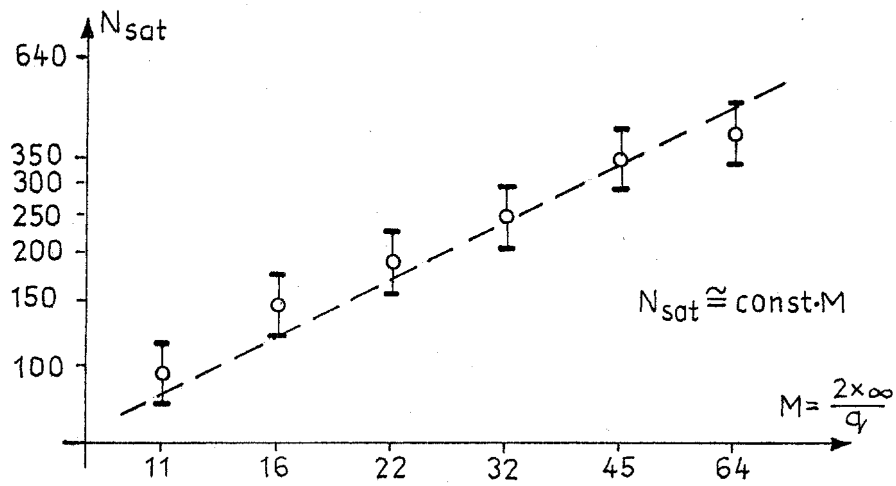

Figure 5: Approximate limit of validity of the noise model

The beginning point (N_{sat}) has been defined as the point where the theoretically calculated standard deviation (continuous lines in *Fig. 4*) equals the experimentally obtained saturated value (dotted lines). As it may be expected, the beginning point can be more or less well approximated by

$$N_{\text{sat}} \approx \text{const} \cdot M \, ,$$

since the increase of M increases approximately linearly the 'frequency' of the quantization error (double as much quantum levels yield approximately double as much 'sawteeth'), and so the quantized signal can be approximately described by a growing number of samples.

Finally simulation has been performed for examining the effect of the quantization of the coefficients a_i. The result is shown in *Fig. 6* and *Fig. 7.*

The simulation shows very low sensitivity for the quantization of the $\{a_i\}$ series. In *Fig. 6* the broken line represents the theoretical standard deviation according to (3) – as if the series $\{a_i\}$ were not quantized. Experimental results seem to be much better approximated by this line than by the theoretically calculated continuous one.

In *Fig. 7* the simulation results are remarkably smaller than it is expected from the noise model:

$$\text{var}\,\{\hat{x}_\infty\} \ll \sum x_i^2 \frac{q_a^2}{12} = \frac{\sum x_i^2}{N} \left(\frac{2}{M_a}\right)^2 \frac{1}{12N} \, .$$

The variance is small because the values a_i do not change from measurement to measurement, so the error of $\hat{x}_\infty$ is not independent in different runs. But since the bias is small as well, we may state that the error of $\hat{x}_\infty$ is scarcely sensitive for small alterations of the optimal series $\{a_i\}$.

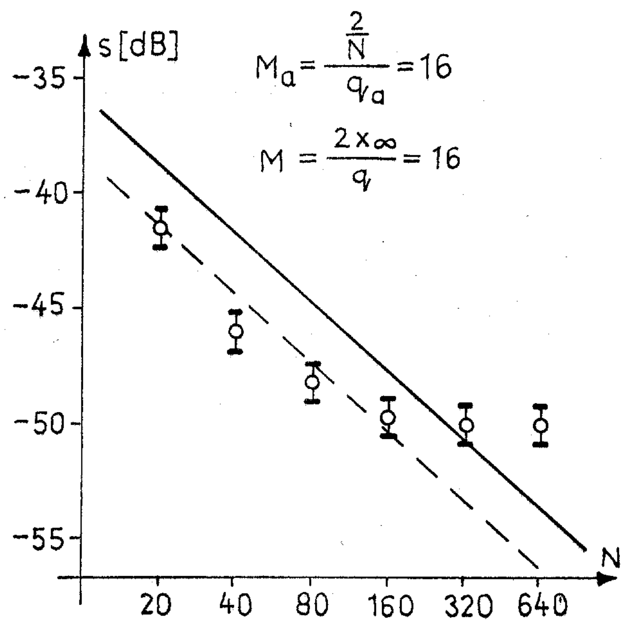

Figure 6: The effect of quantizing $\{a_i\}$ and $x(t)$

4 Realization of the Algorithm

On the basis of the above results we can conclude the following. Since a multiply-and-add operation with 8 bits when multiplying, and 16 bits when adding can easily be performed by a standard 8-bit microprocessor within 1 ms, the weighted averaging of 1000 data using a microprocessor is realistic within 1 second. Since 99% confidence

$$|e_{\max}| < 3\sigma = 3\sqrt{\sum a_i^2}\frac{q}{\sqrt{12}} \, ,$$

and for static measurements the error is limited by $q/2$, the gain when using the algorithm (2) is:

$$g = \frac{\frac{q}{2}}{3\sigma} = 0.54\sqrt{N} \, ,$$

and for $N = 1000$

$$g = 17.1 \Longrightarrow 4.1 \text{ bits} \, .$$

In the simulated case when $T_m = 0.15$ s, with real-time measurements a gain of

$$g = 6.62 \Longrightarrow 2.7 \text{ bits}$$

can be reached. This means that these 2.7 bits can be simply spared at the A/D converter.

Here we must point to an important factor. The theoretical results and simulations have been performed assuming exact quantization levels. The use of less-bit A/D converters generally means less exact quantization levels, and this may give rise to new errors. We expect that if a great portion of the A/D characteristics is used (see e.g. the signal of *Fig. 3*)

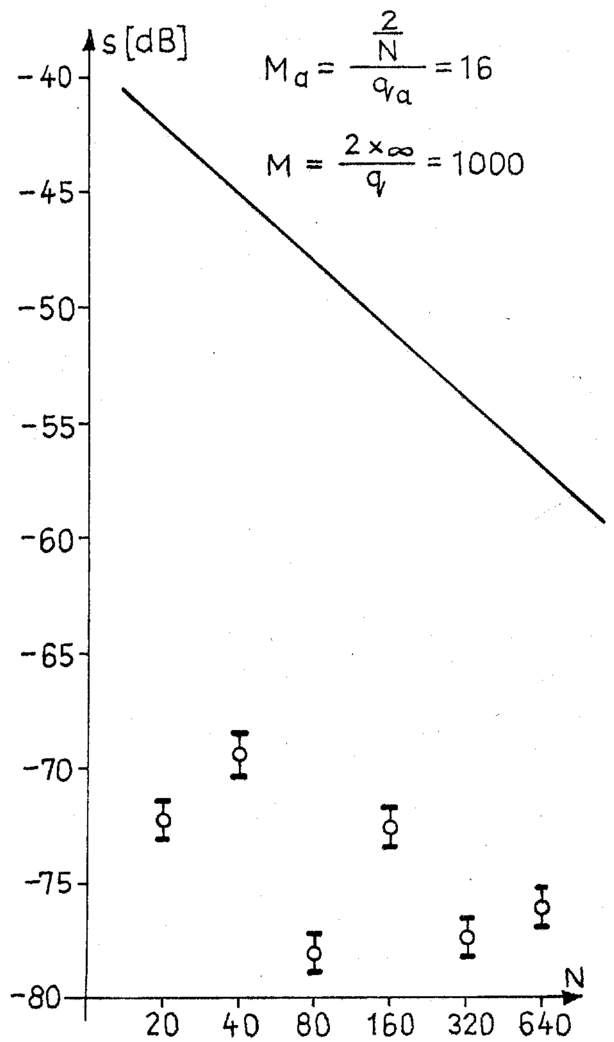

Figure 7: The effect of quantizing only $\{a_i\}$

the differential nonlinearities of the A/D converter are averaged, and still a considerable gain can be achieved.

Finally we state that the above considerations are naturally similar in the case of every algorithm which contains some type of averaging like (2).

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