Errors Introduced by Digital Power/Energy Measurements*

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Abstract

In the paper models are presented and expressions are derived to investigate the effect of sampling and quantization when performing digital power/energy measurements. First a simple and useful model for the comparison of continuous and sampled measurements is given. Then the variance is calculated on the basis of the noise model of quantization, the validity conditions of the model are treated in detail, and an appropriate form of dithering is suggested.
Keywords: Digital wattmeter, dither, quantized data.

1 Introduction

Nowadays power/energy measurements at line frequency are generally performed by means of the good old Ferraris-type dynamometer. This is because of its reliability and low price. However, the accuracy of these instruments is limited: though e.g. 0.25% error limit can be realized, this already requires a controlled environment [1]. Another drawback is the frequency dependence (see *Fig. 1* [2], [3]). Since higher harmonics are already significantly present not only in the current, but in the line voltage waveform as well [2], there is an increasing demand for measuring the real, that is, nonsinusoidal energy as accurately as possible.

A promising though still much more expensive solution for the above problems is the use of electronic wattmeters and watt-hour-meters. Different approaches are investigated currently, like pulse-width modulation, the use of an analog multiplier, digital calculation etc. (see [3]–[5]).

In this paper we shall treat the last one. We shall try to formulate appropriate models to describe the effect of digital processing. First a simple model for the comparison of continuous and sampled power/energy measurements is presented, then quantization is dealt with.

2 A Simple Model of Power Measurements

Results concerning power measurements based on regularly spaced sample pairs can be found in the literature [1], [5], [6]. These investigations usually deal with the deviation of the result from the theoretical value of the power:

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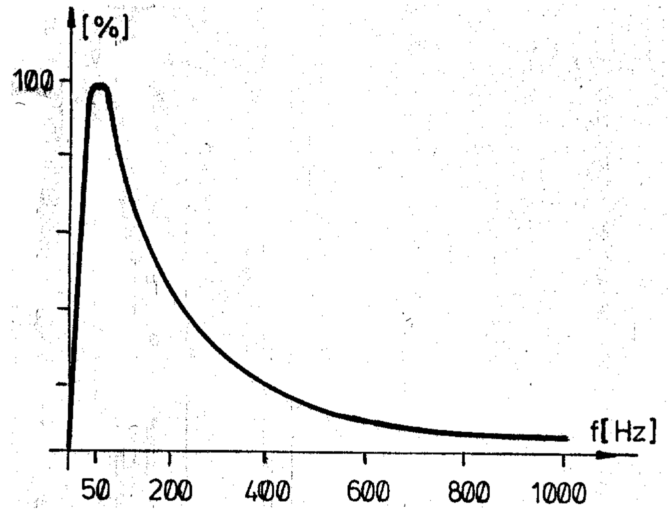

Figure 1: Frequency response of a dynamometer in proportion to the 50 Hz-value

$$P = \frac{1}{T}\int_0^T u(t)i(t)\,\mathrm{d}t\,,$$

where it is assumed that T contains an integral number of cycles. However, by the aid of an appropriate model the results may also be compared with the continuous case, supposing arbitrary measurement time.

First let us consider pure sinusoidal current and voltage, and introduce the following notations:

$$u(t) = U_p\cos(\omega t + \varphi_u)\,; \quad i(t) = I_p\cos(\omega t + \varphi_i)\,; \quad \varphi = \varphi_u - \varphi_i\,.$$

The average power in a given measurement time T_m can be expressed as follows:

$$P = \frac{1}{T_m}\int_0^{T_m} u(t)i(t)\,\mathrm{d}t = \frac{1}{N\Delta t}\sum_{n=1}^{N}\int_{t_n-\frac{\Delta t}{2}}^{t_n+\frac{\Delta t}{2}} u(t)i(t)\,\mathrm{d}t\,, \tag{1}$$

where $\Delta t = T_m/N$ (see *Fig. 2*). This P should be compared with the power obtained from the current and voltage samples:

$$P_s = \frac{1}{N}\sum_{n=1}^{N} u(t_n)i(t_n)\,. \tag{2}$$

In order to compare P and P_s first let the expression of the power in the intervals

$$\left(t_n - \frac{\Delta t}{2}, t_n + \frac{\Delta t}{2}\right)$$

be investigated:

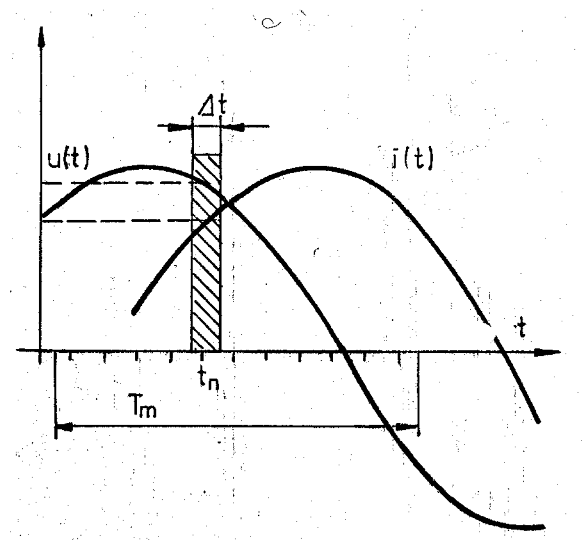

Figure 2: 'piecewise' calculation of the average power

$$
\begin{aligned}
P_\Delta &= \frac{1}{\Delta t}\int_{t_n-\frac{\Delta}{2}}^{t_n+\frac{\Delta}{2}} U_p\cos(\omega t+\varphi_u) I_p \cos(\omega t+\varphi_i)\,\mathrm{d}t = \\
&= \frac{U_p I_p}{2}\left[\cos\varphi + \frac{\sin\omega\Delta t}{\omega\Delta t}\cos(2\omega t_n+\varphi_u+\varphi_i)\right] , \qquad (3)\\
P_{s\Delta} &= u(t_n)i(t_n) = \frac{U_p I_p}{2}[\cos\varphi + \cos(2\omega t_n+\varphi_u+\varphi_i)] . \qquad (4)
\end{aligned}
$$

Comparing P_Δ and $P_{s\Delta}$ we may observe that the only difference is the factor $\frac{\sin\omega t}{\omega\Delta t}$ in (2.3). The same holds for P and P_s:

$$
\begin{aligned}
P &= \frac{U_p I_p}{2}\left[\cos\varphi + \frac{\sin\omega\Delta t}{\omega\Delta t}\frac{1}{N}\sum_{n=1}^{N}\cos(2\omega t_n+\varphi_u+\varphi_i)\right] , \qquad (5)\\
P_s &= \frac{U_p I_p}{2}\left[\cos\varphi + \frac{1}{N}\sum_{n=1}^{N}\cos(2\omega t_n+\varphi_u+\varphi_i)\right] . \qquad (6)
\end{aligned}
$$

Expression (2.6) is discussed extensively in the literature (see e.g. [5], [6]). Here we point out some relations to expression (2.5). If $T_m = MT_p = M\frac{2\pi}{\omega}$ (M is integer), that is, an integral number of cycles is measured, and $\frac{2M}{N}$ is not integer (see Dix, [5]), then both sums in (2.5) and (2.6) disappear, and there is no error introduced by sampling and finite measurement time. If neither M nor $\frac{2M}{N}$ is integer, both in (2.5) and in (2.6) errors remain which tend to zero in the order of $\frac{1}{N}$.

In the case when M and $\frac{2M}{N}$ are integer, the second term in (2.5) disappears because $\omega\Delta t = \frac{\omega T_m}{N} = \frac{2M}{N}\pi$, but in (2.6) generally an error remains. This corresponds to an 'unlucky' sampling with too low sampling frequency (see e.g. *Fig. 3*). It can be seen that the fulfillment of $2M < N$ or $\Delta t < \frac{T_p}{2}$ (more than 1 sample from a cycle of the instantaneous power!) is enough to avoid such situations.

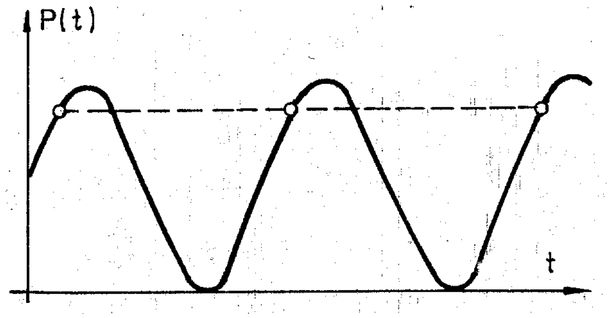

Figure 3: 'unlucky' sampling

Provided that $\Delta t < \frac{T_p}{2}$ (now T_m/T_p may not be integer), the second terms in (2.5) and (2.6) represent the errors which similarly tend towards zero if N is increased, that is, the error in average power measurements vanishes in both cases with an increasing measurement time.

For the investigation of the effect of the harmonics a model similar to the above one may be used. Omitting the calculations, for the integers $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$.

$$\begin{aligned} P_{skl} &= \frac{U_{pk} I_{pl}}{2} \left[\frac{1}{N} \sum_{n=1}^{N} ((k-1)\omega t_n + \varphi_{uk} - \varphi_{il}) \right. \\ &\qquad \left. + \frac{1}{N} \sum_{n=1}^{N} \cos((k+1)\omega t_n + \varphi_{uk} + \varphi_{il}) \right], \qquad (7) \\ P_{kl} &= \frac{U_{pk} I_{pl}}{2} \left[\frac{\sin\left(\frac{k-1}{2}\omega \Delta t\right)}{\frac{k-1}{2}\omega \Delta t} \frac{1}{N} \sum_{n=1}^{N} \cos((k-1)\omega t_n + \varphi_{uk} - \varphi_{il}) \right. \\ &\qquad \left. + \frac{\sin\left(\frac{k+1}{2}\omega \Delta t\right)}{\frac{k+1}{2}\omega \Delta t} \frac{1}{N} \sum_{n=1}^{N} \cos((k+1)\omega t_n + \varphi_{uk} + \varphi_{il}) \right]. \qquad (8) \end{aligned}$$

The considerations may be similar to the above one. It is easy to see that the highest harmonic in the instantaneous power is the one produced by $k_{\max}$ and $l_{\max}$, and the condition

$$\Delta t < \frac{T_p}{k_{\max} + l_{\max}} = \frac{2\pi}{\omega(k_{\max} + l_{\max})}$$

provides that by increasing N, the error tends to zero.

3 The Effect of Quantization

Let us suppose for the moment that – using several-bit A/D converters – the white noise model may be used to describe the quantization error [7], [8]. This means that quantization is modeled by means of an additive white noise of uniform distribution between $\left(-\frac{q}{2}, \frac{q}{2}\right)$:

$$u_q(t_n) = u(t_n) + n_u(t_n)\,, \quad i_q(t_n) = i(t_n) + n_i(t_n)\,. \tag{9}$$

With this model

$$P_{sq} = \frac{1}{N}\sum_{n=1}^{N} Bigl(P_{s\Delta}(t_n) + u(t_n)n_i(t_n) + i(t_n)n_u(t_n) + n_u(t_n)n_i(t_n)\Big)\,.$$

Since the noises are white, independent, and they have zero mean:

$$\begin{aligned}
E\{P_{sq}\} &= P_s\,,\\
\operatorname{var}\{P_{sq}\} &= E\left\{\left[\frac{1}{N}\sum_{n=1}^{N} u(t_n)n_i(t_n) + i(t_n)n_u(t_n) + n_i(t_n)n_u(t_n)\right]^2\right\} =\\
&= \frac{q_i^2}{12N}\frac{1}{N}\sum_{n=1}^{N} u^2(t_n) + \frac{q_u^2}{12N}\frac{1}{N}\sum_{n=1}^{N} i^2(t_n) + \frac{q_u^2}{12N}\frac{q_i^2}{12N}\\
&\approx \frac{1}{N}\left[\frac{q_i^2}{12}U_{\text{eff}}^2 + \frac{q_u^2}{12}I_{\text{eff}}^2 + \frac{1}{N}\frac{q_u^2}{12}\frac{q_i^2}{12}\right]\,.
\end{aligned} \tag{10}$$

Equation (3.2) makes possible to estimate the variance caused by quantization. It suggests that increasing N the error caused by quantization vanishes. Thus, the main problem of digital measurements seems to be solved: the errors caused by sampling and quantization can be estimated and tend towards zero as $\mathcal{O}(1/N)$.

Unfortunately, the situation is not that simple. In the above considerations about quantization the following two assumptions were made, which still must be checked:

1. The A/D converter can be modeled by an ideal uniform quantizer, that is, the quantum levels are exact;

2. The white noise model can be applied.

In the following we shall deal with the A/D conversion of the current; the extension to the 'voltage channel' is straightforward.

3.1 The bias of the quantum levels

For real A/D converters the ideal uniform quantizer is not a completely realistic model. The quantum levels are guaranteed only not to be further than e.g. 1 LSB from the ideal characteristics; they not only have some variance – this could be eliminated by means of averaging – but each of them may have bias as well.

To overcome the above difficulties, it is common to use several-bit A/D converters which provide low bias. Let us take an example for this. Suppose that the voltage in a system does not deviate much from its nominal value, but the amplitude of the current changes from 5% to 250% of its nominal value. We should like to measure power with an accuracy of 0.5%. Let $\cos\varphi$ be 1.

Since the whole domain of the maximum current must be covered by the A/D converter, the following inequality must hold:

$$2.5 \cdot 2I_{\text{pnom}} < 2^B q_i\,, \tag{11}$$

where B is the number of bits and q_i is the quantum size. The requirement of accuracy can be met if

$$b_i < 1\,LSB = q_i < 0.005 \cdot 0.05 I_{\mathrm{effnom}} \,. \tag{12}$$

From (3.3) and (3.4)

$$B > 14.8 \Longrightarrow 15 \text{ bits} \,. \tag{13}$$

However, this result is too pessimistic. Once several quantum levels are used (in the example with 15 bits even for the 5% current amplitude more than 655 levels are used), the biases of the different quantum levels generally work against each other. This means that we may use cheaper A/D converters.

A rough estimation can be obtained by assuming that the biases of the different quantum levels are independent. The resulting distortion can be expressed by a weighted sum: the weighting function is the p.d.f of the voltage. After some calculations a limit of the average bias can be obtained using the 95% confidence level of the normal distribution ($\pm 2\sigma$):

$$b_i \approx \frac{2q_i}{\sqrt{K_{\min}}} = \frac{2q_i}{\sqrt{2^B \frac{0.05}{2.5}}} \,, \tag{14}$$

where $K_{\min}$ is the number of the crossed quantum levels in the case of the minimal I_p. From (3.6) and (3.3):

$$B > 12.4 \Longrightarrow 13 \text{ bits} \,. \tag{15}$$

It is clear that the averaging of possibly many quantum levels is more favorable. This can be provided by *dithering.*

Dithering means adding to the A/D input a zero-mean, independent, auxiliary signal, which does not disturb strongly the estimation, but effectively 'moves' the signal, utilizing an extensive part of the A/D characteristics.

The choice of such a dither is rather obvious in our case: a large-amplitude, independent sine wave e.g. of frequency $\sim 1.5 \times 50$ Hz and of amplitude $I_{pd} \sim 2^B q_i/4$ (thus allowing $2^B q_i/4$ for I_p as well) will have virtually no effect on the estimator if the sampling frequency and the number of samples are appropriately chosen. With this dither

$$B > 10.85 \Longrightarrow 11 \text{ bits} \,, \tag{16}$$

and this is already a much more acceptable condition.

If the biases of quantum levels may not be assumed to be independent, that is, they have some structure, identification or self-calibration methods can often be applied to get rid of the effect of them.

3.2 The validity of the white noise model

The quantization noise is usually assumed to be independent, uniform and white. The condition to have an approximately white spectrum is given in [8]:

$$f_s < 3\frac{I_p}{q_i} f_1 \,, \tag{17}$$

where f_s is the sampling frequency. Condition (3.9) is generally fulfilled. However, it is easy to see that in the case of synchronized sampling the samples of the quantization error are not independent any more. The problem may be solved by using another, broad-band, Gaussian dither of small ($2\sigma < q_i$) amplitude: the increase of the variance can be described replacing $q_1^2/12$ by $(\sigma^2 + q_i^2/12)$ in (3.2).

In our case independence and uniform distribution are not by all means needed. The variance caused by quantization is approximately $q_i^2/12$, and this is enough to have an estimate of the resulting variance (see expression 3.2), but the bias caused by the quantization of the sine wave is still to be calculated. This is a rather complicated task, and it can be done by numerical calculations. However, the introduction of the above Gaussian dither simplifies the situation. Quantized Gaussian signals have very small bias [7]:

$$\max_{\mu_x}\{E\{x_q - x\}\} \approx \frac{q}{\pi} e^{-2\pi^2 \frac{\sigma_x^2}{q^2}} ,$$

which is less than $0.01q$ if $\sigma_x > 0.42q$. Thus the use of the Gaussian dither can provide the validity of the noise model.

Summarizing the considerations about quantization, a combined (sine+Gaussian) dither may be suggested for the 'current-channel', and a small Gaussian dither for the 'voltage-channel'.

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