IEEE Instrumentation and Measurement Technology Conference
Brussels, Belgium, June 4–6, 1996. pp. 14–19.

On the Equivalence of z-domain and s-domain Models in System Identification

István Kollár*, Gene Franklin** and Rik Pintelon***

*Department of Measurement and Instrument Engineering
Technical University of Budapest
Budapest, Műegyetem rkp. 9, Hungary, H-1521
Phone: + 36 1 463-1774, Fax: + 36 1 463-4112
Email: kollar@mmt.bme.hu

**Information Systems Laboratory
Department of Electrical Engineering
Stanford University
Stanford, CA 94305-4055, USA
Phone: + 1 415 723-4837, Fax: + 1 415 723-8473
Email: franklin@isl.stanford.edu

***Department ELEC
Vrije Universiteit Brussel
Pleinlaan 2, Brussels, Belgium, B-1050
Phone: + 32 2 629-2944, Fax: + 32 2 629-2850
Email: rpintel@vnet3.vub.ac.be

Abstract — **The step-invariant or ZOH-transformation of discrete-time transfer function models to continuous time is discussed. It is shown that negative real poles which are commonly considered as not transformable, can be transformed to poles on the aliasing margin. Thus a generalized equivalent can be determined, whose ZOH-transform is the original system. Numerical algorithms are also discussed, and examples are shown.**

Keywords — **System identification, discrete-to-continuous time conversion, zero-order hold, ZOH, indirect identification, step invariant transform.**

I. Introduction

Identification of continuous-time linear systems can be done either in time domain or in frequency domain. Time domain methods usually determine a discrete-time (z-domain) model of the system, while frequency domain methods can identify either a discrete-time (z-domain) or a continuous-time (s-domain) model. For proper identification, the measurement has to meet certain conditions that are different for z-domain and s-domain [1]. Therefore, the measurement arrangement determines which is the proper model to identify from the measured data.

When a z-domain model has to be identified, the usual underlying assumption is that the excitation signal is constant between the sampling instants (zero-order hold or ZOH assumption, see Fig. 1), and the output signal is sampled without band-limiting it by an anti-aliasing filter [1], [2]. If these assumptions are fulfilled, the s-domain rational transfer function of the system uniquely determines a corresponding z-domain model through the so-called step invariant (or ZOH-) transform [3], [4], [5]:

$$H(z) = \mathrm{ZOH}\left\{H(s)\right\} = (1-z^{-1})\mathcal{Z}\left\{\frac{H(s)}{s}\right\}, \quad (1)$$

where $\mathcal{Z}\left\{\frac{H(s)}{s}\right\}$ is the shorthand for $\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}\right\}$, and $\mathcal{Z}\{\cdot\}$, $\mathcal{L}\{\cdot\}$ denote the z- and Laplace transforms, respectively.

It is clear that the transform in (1) is linear. It can be shown [4], [3], [5] that the ZOH-transform of a first-order system is

$$\mathrm{ZOH}\left\{\frac{a}{s+a}\right\} = \frac{1-e^{-aT}}{z-e^{-aT}}, \quad (2)$$

that is, the s-domain pole $s_1 = -a$ is transformed to a z-domain pole

$$z_1 = e^{-aT} = e^{s_1 T}, \quad (3)$$

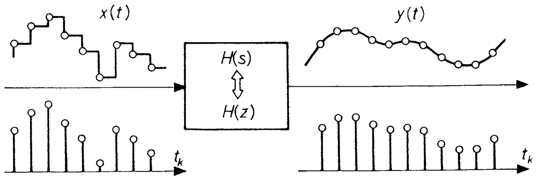

Fig. 1. ZOH assumption

where T is the sampling interval (Fig. 2). Similarly, multiple s-domain poles are transformed to multiple z-domain poles through the same mapping. Therefore, since rational transfer functions can be expanded into partial fractions with the poles in the denominators, the mapping of all of the poles is known. The zeros cannot be related to each other by a similarly simple mapping [6].

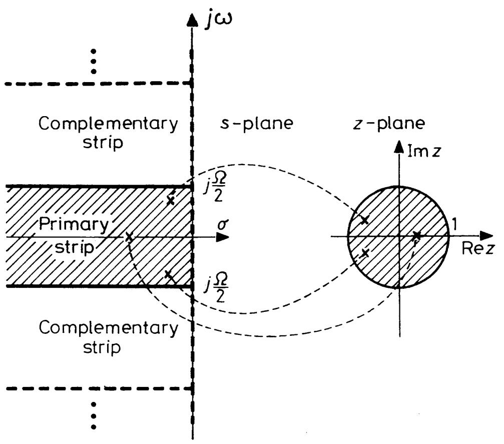

Fig. 2. Mapping between s-domain and z-domain

The equivalent z-domain model is estimated by identification methods. In indirect identification, when the s-domain model of the system is sought, the question is whether the z-domain model uniquely determines the s-domain model or not.

For the case when there is no negative real pole in the z-domain transfer function, furthermore it may be assumed that the sampling radian frequency Ω is larger than double of the largest imaginary part of the s-domain poles and there is no extra delay, the $z \to s$ transformation is unique and straightforward. The calculation can be done by inversion of the $s \to z$ pole mapping given by (3), but dealing with the partial fractions is cumbersome. Therefore, $z \to s$ transformation is usually done by taking the logarithm of the state-space transition matrix [7], [8]. Another possibility is to directly determine the transfer function coefficients by fitting of time domain sequences [10].

II. Dealing with a Negative Real Pole in the z-domain

Neither of the above described $z \to s$ transform methods work when there is a negative real pole. The problem is that the logarithm of a negative number is complex: $\log(-a) = \log(a) + j(\pi \pm 2k\pi)$, $k = 0, \pm 1, \pm 2, \dots$ The transformation is not unique. But even if we select a certain leaf of the complex logarithm function, we end up with a complex pole without a complex conjugate pair. This has no meaning since we are looking for a real-coefficient s-domain rational transfer function. This is the reason why it is common conviction that negative real poles cannot be transformed, so in such cases there is no s-domain equivalent [5], [11], unless there is a negative real pole pair [12]. Usual methods generate a warning message [7], [8] in the case of negative real poles, and take the real part of the complex matrix logarithm. The error introduced depends on the importance of the negative real poles. When they are far from the unit circle, their effect is small, and the resulting transfer function may be acceptable. The ZOH-transform of the so obtained s-domain model is however not the original z-domain one.

The problem described above is two-fold.

- Even if we decide for the central leaf, we cannot determine which of the values $\log(a) \pm j\pi$ should be chosen;
- There is an s-domain pole without a complex conjugate pair.

At this point we may speculate whether we can assign an equivalent *pole pair* to the single z-domain one. This would only be possible if a *double* negative pole was treated. Let us assume that the second pole is cancelled by a zero at the same position. This can be examined as a limiting case for $\varphi \to \pi/T - 0$ of the following system:

$$\begin{aligned} H_\varphi(s) &= \frac{\alpha + j\varphi}{s + (\alpha + j\varphi)} + \frac{\alpha - j\varphi}{s + (\alpha - j\varphi)} \\ &= 2\frac{\alpha s + \alpha^2 + \varphi^2}{(s+\alpha)^2 + \varphi^2} \\ &\underset{\varphi \to \pi/T - 0}{\longrightarrow} \quad 2\frac{\alpha s + \alpha^2 + (\pi/T)^2}{(s+\alpha)^2 + (\pi/T)^2} \,. \end{aligned} \tag{4}$$

By comparing (2) and (4) it is easy to see that the ZOH-equivalent is, using $a = \alpha + j\varphi$,

$$\begin{aligned} H_\varphi(z) &= \text{ZOH}\left\{H_\varphi(s)\right\} \\ &= \frac{1 - e^{-(\alpha + j\varphi)T}}{z - e^{-(\alpha + j\varphi)T}} + \frac{1 - e^{-(\alpha - j\varphi)T}}{z - e^{-(\alpha - j\varphi)T}} \\ &= 2\frac{z(1 - e^{-\alpha T}\cos(\varphi T)) + e^{-2\alpha T} - e^{-\alpha T}\cos(\varphi T)}{z^2 - 2z e^{-\alpha T}\cos(\varphi T) + e^{-2\alpha T}} \,. \end{aligned} \tag{5}$$

For $\varphi \rightarrow \pi/T$ the two poles and the zero all become equal to $e^{-\alpha T}$, pole-zero cancelation occurs, and we obtain a single negative real pole:

$$\begin{aligned} H(z) &= \lim_{\varphi \rightarrow \pi/T-0} H_{\varphi}(z) \\ &= 2\frac{z(1+e^{-\alpha T})+e^{-2\alpha T}+e^{-\alpha T}}{z^2+2ze^{-\alpha T}+e^{-2\alpha t}} \\ &= 2\frac{1+e^{-\alpha T}}{z+e^{-\alpha T}}\,. \end{aligned} \tag{6}$$

By these calculations we have shown that there is a possible s-domain equivalent of a real negative z-domain pole, at least if s-domain poles with imaginary parts equal to the Nyquist radian frequency are allowed. Equation (6) is the ZOH transform of the limiting value of (4). However, since in this case the poles alias, strange phenomena can be expected. Indeed, the ZOH-transform of the following system is zero:

$$\begin{aligned} &\mathrm{ZOH}\left\{\frac{b_1 s}{s^2+2\alpha s+\alpha^2+(\pi/T)^2}\right\} \\ &= \mathrm{ZOH}\left\{\frac{\frac{b_1(-(\alpha+j\pi/T))}{-2j\pi/T}}{s+(\alpha+j\pi/T)}+\frac{\frac{b_1(-(\alpha-j\pi/T))}{2j\pi/T}}{s+(\alpha-j\pi/T)}\right\} \\ &= \left(\frac{b_1 T}{2j\pi}\right)\frac{1+e^{-\alpha T}}{z+e^{-\alpha T}}-\left(\frac{b_1 T}{2j\pi}\right)\frac{1+e^{-\alpha T}}{z+e^{-\alpha T}} \\ &= 0\,. \end{aligned} \tag{7}$$

In the before last step we made use of (2).

This means that such terms have no effect at all on the z-domain equivalent! This phenomenon is known as *hidden oscillation* in control [4], [5]. A damped sine wave is sampled at exactly the zero crossings (Fig. 3).

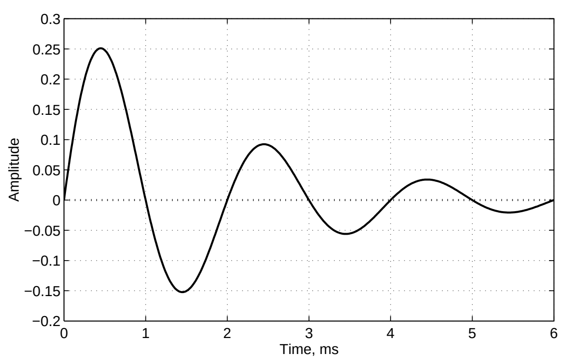

Fig. 3. Step response of the hidden term (8) for $b_1 = 1\,\mathrm{ms}$, $\alpha = 0.5\,\mathrm{kHz}$, $T = 1\,\mathrm{ms}$

There is a whole set of s-domain equivalents for the z-domain transfer function which are equal at the sampling instants (Fig. 4). If a discrete system has a negative real pole at $z_1 = -e^{-\alpha T}$, the s-domain equivalent is undetermined up to an additive term

$$H_{1h}(s) = \frac{b_1 s}{(s+\alpha)^2+(\pi/T)^2}\,, \tag{8}$$

with arbitrary b_1. In other words, the poles of the s-domain system are fully determined, the zeros are not. The zero of the ZOH-equivalent of a first-order z-domain system can be anywhere. Zeros of the equivalents of higher-order z-domain systems cannot be anywhere, since the degree of freedom is only one, but none of the zeros has a determined position.

These s-domain systems are all exact equivalents of the z-domain one in the sense that for ZOH-input their outputs at the sampling instants are the same as that of the discrete system, but they differ in their inter-sample behavior. That is, the discrete-time transfer function does not uniquely determine the inter-sample behavior when a negative real pole is present (Fig. 4).

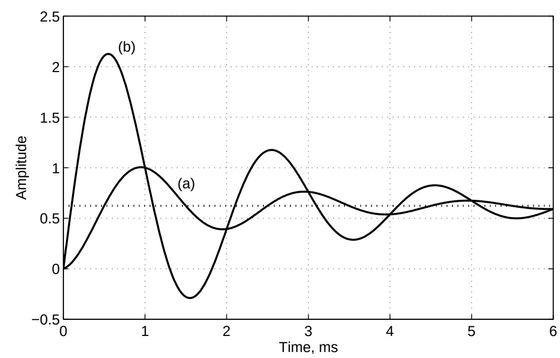

Fig. 4. Step responses of equivalent s-domain systems with a single pole pair (a) the "minimal system $H_{1min}(s)$, with $\zeta = 1.65$, $T = 1\,\mathrm{ms}$ (9); (b) with modified numerator with $b_1 = b_0 * T$.

Here the question may arise whether it is worth dealing with such an obscure case at all. Indeed, when it is possible to make a new measurement with a higher sampling frequency, and/or possibly better SNR, it should be done, and the new identification session will tell whether there is a pole pair where we assigned it or not. But when an available z-domain model must be evaluated, there is no better way than to accept this situation and obtain the best achievable result. The example in Section IV illustrates that this yields a reasonable model.

The only way we can suggest for the choice of the s-domain equivalent is to use some kind of parsimony principle. For example, we can choose the s-domain model which minimizes the inter-sample oscillations. Let us require that there is no sine term in the step response. Therefore, we suggest to assign the term

$$\frac{1}{z+\zeta} \Rightarrow H_{1min}(s)$$

$$= \frac{1}{1+\zeta} \cdot \frac{s\alpha + \alpha^2 + (\pi/T)^2}{(s+\alpha)^2 + (\pi/T)^2} \qquad (9)$$

(see Appendix, Eq. (18)) to the single negative real poles (Fig. 4a), where $\alpha = -(\ln\zeta)/T$.

III. Numerical Techniques

We have seen that a whole set of s-domain transfer functions can be assigned to the z-domain one with a negative real pole. Such an ambiguity usually manifests itself in bad numerical conditioning of the transformation. Indeed, for negative real poles conditioning is very bad. The only effective remedy is to eliminate the cause of bad conditioning. We can determine the partial fractions belonging to the negative real poles, subtract them from the transfer function, and transform them separately (see (9) for a single pole, and see below for multiple poles). Although subtraction of the partial fractions can also be badly conditioned, this is still much more reliable than to blindly take the logarithm, and for moderate system orders it gives sensible models.

Another possibility would be to "tell" the logarithm what to do. Adding a pole and a zero with the same value as the existing negative real pole does not help conditioning. But there is a simple recipe which yields a very simple algorithm: whenever there is a negative real pole, add a zero of the same value to the numerator, and replace the pole with a *pole pair* with the same real part and tiny imaginary parts, e.g. somewhat larger than the square root of the numerical precision of the given computer times the real part.

In the Appendix the direct calculation of the s-domain equivalents is illustrated for single and double poles. For poles with higher multiplicity the calculations are too complicated. Therefore, we need a roundabout, possibly in the form of a calculation algorithm.

We may recognize that although the $z \to s$ transform is badly conditioned, the $s \to z$ one is not. Therefore, if a term $1/(z+\zeta)^n$ is to be transformed, we may use Matlab or another CAD package to determine the z-domain equivalents of

$$H_{nk}(s) = \frac{b_k s^k}{\left((s+\alpha)^2 + (\pi/T)^2\right)^n}, \quad k = 0, 1, \ldots, n-1. \qquad (10)$$

A little manipulation is necessary since the calculated z-domain equivalents contain n cancelling pole/zero pairs each, which must be factored out, but since the orders are moderate, this can be done with acceptable accuracy.

We can then express the z-domain fraction to be transformed by the transforms of the $H_{nk}(s)$'s as

$$1/(z+\zeta)^n = \sum_{k=0}^{n-1} c_k H_{nk}(z), \qquad (11)$$

and obtain the result as

$$H(s) = \sum_{k=0}^{n-1} c_k H_{nk}(s). \qquad (12)$$

This procedure is well defined, and can be easily programmed. The only problem arises because of the hidden terms which correspond to step responses of the form $e^{-\alpha t} t^m \sin(\pi t/T)$, $m = 0, 1, \ldots, n-1$. Since their z-domain equivalents are identically zero, the solution of (11) is not unique. Every solution yields an equivalent s-domain function, so any one of them is acceptable. The parsimony principle can be applied again, suppressing these hidden terms. For $m = 0, 1$ the hidden terms are as follows [13, p. 54, Eq. 7.1; p. 220, Eq. 2.41]:

$$s\mathcal{L}\left\{e^{-\alpha t}\sin(\pi t/T)\right\} = \frac{(\pi/T)s}{(s+\alpha)^2 + (\pi/T)^2},$$
$$s\mathcal{L}\left\{e^{-\alpha t} t \sin(\pi t/T)\right\} = \frac{2(\pi/T)s(s+\alpha)}{\left((s+\alpha)^2 + (\pi/T)^2\right)^2}, \qquad (13)$$

with $\alpha = -(\ln\zeta)/T$. The general form of the hidden terms is rather complicated [13, p. 55, Eq. 7.15]:

$$s\mathcal{L}\left\{e^{-\alpha t} t^m \sin(\pi t/T)\right\} = \frac{s \cdot m! \sin\left((m+1)\arctan\frac{\pi}{T(s+\alpha)}\right)}{\left(\sqrt{(s+\alpha)^2 + (\pi/T)^2}\right)^{m+1}}, \qquad (14)$$

This can be expressed as a rational form for every m, using

$$\sin(n\gamma) = \sum_{k=1}^{\lfloor n/2+1 \rfloor} (-1)^{k+1} \binom{n}{2k-1} \cos^{n-(2k-1)}\gamma \sin^{2k-1}\gamma \qquad (15)$$

and

$$\sin\gamma = \frac{\tan\gamma}{\sqrt{1+\tan^2\gamma}}, \quad \cos\gamma = \frac{1}{\sqrt{1+\tan^2\gamma}}, \qquad (16)$$

valid for $|\gamma| < \pi/2$, but it is not an easy job. Instead of dealing with this formula, we accept the least squares solution of equation (11). This does not necessarily eliminate the sine-like terms, but it will allow only a small contribution of them.

IV. Numerical Example

An interesting example for indirect identification is presented in [11]. The authors identified the fuel flow to shaft speed dynamics of a gas turbine. They obtained z-domain models as given in Table I. They observed that the fit was good for both the 1/2 and the 2/3 systems, but since there was a negative real pole in the 1/2 one, they discarded it, and used the 2/3 one.

	z-domain		s-domain equivalent	
Order	Zeros	Poles	Zeros	Poles
0/1		0.9002		-0.5256
1/2	0.1684	0.9101	-2.7153	-0.4712
		-0.4830		???
2/3	0.0699	0.9098	$-8.15\pm j7.41$	-0.4728
	-0.4245	$-0.49+j0.39$		$-2.35+j12.33$
		$-0.49-j0.39$		$-2.35-j12.33$
	s-domain equivalent with new method			
1/2	0.1684	0.9101	-3.4942	-0.4712
		-0.4830	-3.3955	-3.6381
			$\pm j14.386$	$\pm j15.7052$

TABLE I

z-DOMAIN POLES AND s-DOMAIN EQUIVALENTS (AFTER [11])

We have used the procedure described in this paper to obtain an s-domain model corresponding to the 1/2 z-domain one. Figures 5-8 show the Bode diagrams of the s-domain equivalents. Figures 5-7 were calculated using Matlab's `d2cm` function, neglecting the warning for the 1/2 case. The transfer functions are similar, though there are significant differences between the finally accepted 2/3 system, and the other two.

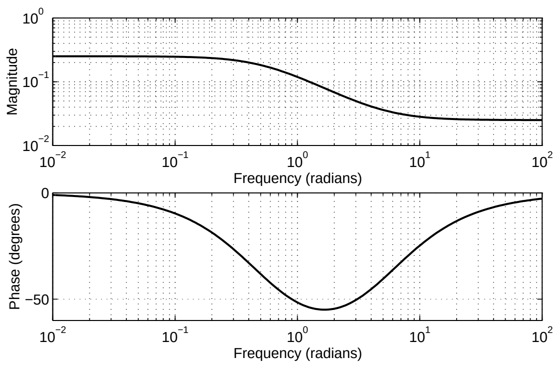

Fig. 5. Transfer function of the equivalent of the 0/1 system, calculated via `d2cm` in Matlab

Figure 8 illustrates the system obtained from the 1/2 model with the new algorithm. The transfer function remarkably resembles to the "good" one in Fig.7. This illustrates that the complex pole pair introduced in the transformation is not just mere fiction, but may account for important parts of the system dynamics.

V. Conclusions

A new step-invariant z-domain to s-domain conversion method has been developed to deal with negative real z-domain poles, an unsolved problem in commercially available CACSD packages. The new method assigns a complex s-domain pole pair at the Nyquist frequency to

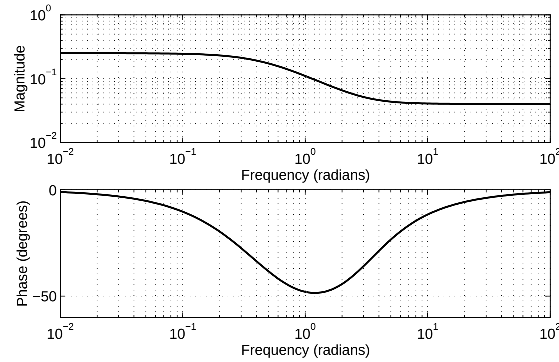

Fig. 6. Transfer function of the equivalent of the 1/2 system, calculated via `d2cm` in Matlab

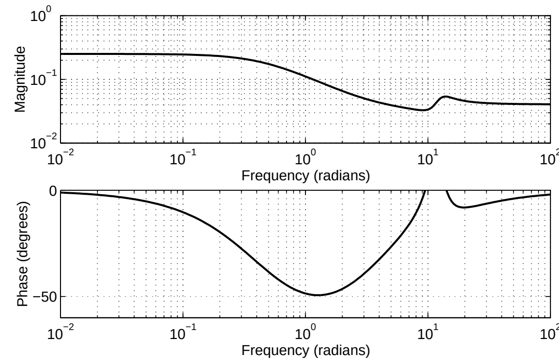

Fig. 7. Transfer function of the equivalent of the 2/3 system, calculated via `d2cm` in Matlab

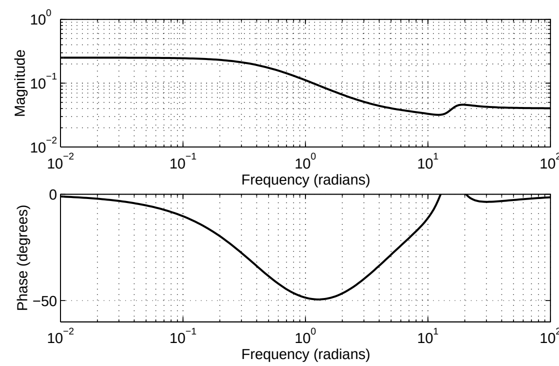

Fig. 8. Transfer function of the equivalent of the 1/2 system, calculated with the proposed algorithm

the system. The conversion does not suffer from bad conditioning, and gives a reasonable s-domain equivalent of the discrete-time model. The order of the s-domain system is is higher than that of the z-domain one, by the number of the negative real poles.

References

[1] J. Schoukens, R. Pintelon, and H. Van hamme, "Identification of linear dynamic systems using piecewise constant ex-

citations: Use, misuse and alternatives," *Automatica*, vol. 30, no. 7, pp. 1153–69, 1994.
[2] L. Ljung, *System Identification: Theory for the User*. Englewood Cliffs, NJ: Prentice-Hall, 1987.
[3] R. H. Middleton and G. C. Goodwin, *Digital Control and Estimation*. Englewood Cliffs, NJ: Prentice-Hall, 1990.
[4] G. F. Franklin, J. D. Powell, and M. L. Workman, *Digital Control of Dynamic Systems*. Reading, MA: Addison-Wesley, 2 ed., 1992.
[5] K. Ogata, *Discrete-Time Control Systems*. Reading, MA: Addison-Wesley, 2 ed., 1995.
[6] K. J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems," *Automatica*, vol. 20, pp. 31–38, Jan 1984.
[7] A. Grace, A. J. Laub, J. N. Little, and C. M. Thompson, *Control System Toolbox V3.0 for Matlab*. Natick, MA: The MathWorks, Inc., 1993.
[8] L. Ljung, *System Identification Toolbox V4.0 for Matlab*. Natick, MA: The MathWorks, Inc., 1995.
[9] I. Kollár, *Frequency Domain System Identification Toolbox V2.0 for Matlab*. Natick, MA: The MathWorks, Inc., 1994.
[10] A. G. Kuznetsov and D. W. Clarke, "Simple numerical algorithms for continuous-to-discrete and discrete-to-continuous conversion of the systems with time delay," in *Sysid'94, 10th IFAC Symposium on System Identification, Copenhagen, July 4-6, 1994. Vol. 2*, pp. 3.633–38, 1994.
[11] C. Evans, D. Rees, L. Jones, and D. Hill, "Time and frequency domain identification of jet engine dynamics: Problems and solutions," in *Sysid'94, 10th IFAC Symposium on System Identification, Copenhagen, July 4-6, 1994. Vol. 2*, pp. 2.243–48, 1994.
[12] N. K. Sinha and G. J. Lastman, "Transformation of discrete-time models," in *Identification of Continuous-Time Systems* (N. K. Sinha and G. P. Rao, eds.), ch. 4, pp. 123–137, Dordrecht: Kluwer Academic Publishers, 1991.
[13] F. Oberhettinger and L. Badii, *Tables of Laplace Transforms*. Berlin, New York: Springer-Verlag, 1973.

Appendix

Determination of the ZOH-equivalents of single and multiple negative real poles

In this Appendix we illustrate on the example of single and double negative real poles how to derive the ZOH-equivalents.

The general procedure consists of inversion of the steps of (1).

$$\begin{aligned} H(z) \quad &\Rightarrow \quad H_m(z) = \frac{z}{z-1}H(z) \Rightarrow h_m(k) \Rightarrow h_m(t) \\ &\Rightarrow \quad H_m(s) \Rightarrow H(s) = sH_m(s)\,. \end{aligned} \tag{17}$$

Single negative real pole

$$\begin{aligned} \frac{1}{z+\zeta} &\Rightarrow \frac{z}{z-1}\cdot\frac{1}{z+\zeta} = z\left(\frac{\frac{1}{1+\zeta}}{z-1} + \frac{\frac{1}{-\zeta-1}}{z+\zeta}\right) \\ &\Rightarrow \frac{1}{1+\zeta}1(k) - \frac{1}{1+\zeta}(-\zeta)^k 1(k) \\ &\Rightarrow \frac{1}{1+\zeta}1(t) - \frac{1}{1+\zeta}e^{-\alpha t}\cos(\pi t/T)1(t) \\ &\Rightarrow \frac{1}{1+\zeta}\left(\frac{1}{s} - \frac{s+\alpha}{(s+\alpha)^2+(\pi/T)^2}\right) \\ &\quad = \frac{1}{1+\zeta}\cdot\frac{s\alpha+\alpha^2+(\pi/T)^2}{s\left((s+\alpha)^2+(\pi/T)^2\right)} \\ &\Rightarrow \frac{1}{1+\zeta}\cdot\frac{s\alpha+\alpha^2+(\pi/T)^2}{(s+\alpha)^2+(\pi/T)^2}\,, \end{aligned} \tag{18}$$

where $1(k)$ and $1(t)$ are the discrete-time and continuous-time unit step functions, respectively, and $\alpha = -(\ln\zeta)/T$.

In the third line we made use of the fact that the alternating sign can be represented by a cosine function. Note that this is the step where an arbitrary term containing a factor $\sin(\pi t/T)$ could also be added.

Double negative real pole

$$\begin{aligned} \frac{1}{(z+\zeta)^2} \quad &\Rightarrow \quad \frac{z}{z-1}\cdot\frac{1}{(z+\zeta)^2} \\ &\quad = z\left(\frac{\frac{1}{1+\zeta}}{(z-1)(z+\zeta)} - \frac{\frac{1}{1+\zeta}}{(z+\zeta)^2}\right)\,. \end{aligned} \tag{19}$$

We have already determined the s-domain equivalent of the first term of (19) in (18). The equivalent of the second term is as follows:

$$\begin{aligned} & z\frac{-1}{1+\zeta}\cdot\frac{1}{(z+\zeta)^2} \\ &\quad \Rightarrow \frac{-1}{1+\zeta}\cdot\frac{k(-\zeta)^k 1(k)}{(-\zeta)} \\ &\quad \Rightarrow \frac{1}{(1+\zeta)\zeta}\cdot\frac{t}{T}e^{-\alpha t}\cos(\pi t/T)1(t) \\ &\quad \Rightarrow \frac{1}{(1+\zeta)\zeta T}\frac{(s+\alpha)^2-(\pi/T)^2}{\left((s+\alpha)^2+(\pi/T)^2\right)^2} \\ &\quad \Rightarrow \frac{1}{(1+\zeta)\zeta T}\frac{s\left((s+\alpha)^2-(\pi/T)^2\right)}{\left((s+\alpha)^2+(\pi/T)^2\right)^2}\,. \end{aligned} \tag{20}$$

We have constructed the s-domain function keeping in mind to have only cosine terms, thus this is a minimum oscillation solution. We combine it with $1/(1+\zeta)$ times (19):

$$\begin{aligned} & \frac{1}{(z+\zeta)^2} \Rightarrow H_{2min}(s) \\ &\quad = \frac{1}{(1+\zeta)^2}\cdot\frac{s\alpha+\alpha^2+(\pi/T)^2}{(s+\alpha)^2+(\pi/T)^2} \\ &\qquad + \frac{1}{(1+\zeta)\zeta T}\cdot\frac{s\left((s+\alpha)^2-(\pi/T)^2\right)}{\left((s+\alpha)^2+(\pi/T)^2\right)^2}\,. \end{aligned} \tag{21}$$

This formula is still usable, but needed significant effort to calculate. For the rare case of real negative poles with higher multiplicity, a simple roundabout can be suggested (see Section III) instead of direct calculation.