If we let $f(t)$ represent the forcing function of the first-order system being analyzed, we can write an expression for the response of the system in terms of the moments of the impulse response and the derivatives of the forcing function. Thus

$$g(t)=\sum_{r=0}^{\infty} f^{(r)}(t)\left\{\frac{r!}{a^{r+1}}-e^{-at}\sum_{n=0}^{r}\frac{D^{(n)}[t^r]}{a^{n+1}}\right\}(-1)^r. \quad (13)$$

Obviously, the number of terms in the series expansion of the above equation is limited by the number of nonzero derivatives of $f(t)$, the forcing function. In the example previously studied where the forcing function was a ramp, the series would contain only two terms since the second- and higher ordered derivatives of $f(t)$ are zero. Applying (13) to the previous example gives

$$g(t)=\sum_{r=0}^{1} f^{(r)}(t)\left\{\frac{r!}{a^{r+1}}-e^{-at}\sum_{n=0}^{1}\frac{D^{(n)}[t^r]}{a^{n+1}}\right\}(-1)^r.$$

Thus

$$g(t)=f^0(t)\left\{\frac{0!}{a}-e^{-at}\left[\frac{f^0(t^0)}{a}+\frac{f'(t^0)}{a^2}\right]\right\}$$

$$-f'(t)\left\{\frac{1!}{a^2}-e^{-at}\left[\frac{f^0(t)}{a}+\frac{f'(t)}{a^2}\right]\right\}$$

$$g(t)=t\left\{\frac{1}{a}-e^{-at}\left[\frac{1}{a}\right]\right\}-(1)\left\{\frac{1}{a^2}-e^{-at}\left[\frac{t}{a}+\frac{1}{a^2}\right]\right\}.$$

Collecting terms gives

$$g(t)=\frac{t}{a}-\frac{1}{a^2}[1-e^{-at}]. \quad (14)$$

A comparison of the expression in (10) obtained using the Laplace transform method with (14) shows that these two equations are in exact agreement for all values of a and t. This is true because the Taylor series expansion is truncated at the point where we obtain the first zero-valued derivative of the forcing function. Therefore, (13) gives an exact solution for any forcing function which is a finite polynomial in t. This includes the unit step, ramp, and parabola.

IV. Conclusions

The "convolution of a series: forcing-moment" method presented in Section II gives excellent results compared with other methods of analysis. Any degree of accuracy desired can be achieved by using more terms in the series represented by (6). The ratio test [2] for series indicates that the series contained in (6) is convergent for all values of a and t. In the example presented in Section II, we obtained agreement within 0.000171 percent of the Laplace solution using only three terms in the series.

The "impulse-moment" method discussed in Section III gave exact results when compared to the Laplace solution in the same example. This is due to the self-limiting nature of the series represented by (13). Exact solutions will be obtained using this method for all polynomials in t of finite order. This includes such widely used forcing functions as the unit step, ramp, and parabola.

Both of the "series convolution" methods are easily programmed on a digital computer. Although the algorithms given are for first-order systems only, the response of higher order systems can be found by decomposing such systems into combinations of first-order systems (by state variable techniques, for example).

Through the study of the time-moments of a system, we can gain insight into the behavior of that system.

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Sensitivity Properties of Resonator-Based Digital Filters

GÁBOR PÉCELI

Abstract —**In this paper the sensitivity properties of the recently introduced common structure for recursive discrete transforms [1] are investigated. This structure is based on digital resonators in a feedback loop. It is shown that this structure has very nice sensitivity properties near to the resonator pole frequencies, and that by locating the resonator poles properly, even the "passivity" of the feedback loop can be easily guaranteed.**

I. Introduction

Recently, a common structure for recursive discrete transforms has been suggested [1], which seems to be suitable to form a (possibly VLSI implemented) common base for every linear filtering-like signal processing operation. The derivation of this structure and its parameters is based on the state-variable formulation and the results of the observer theory [2], while its applicability to FIR and IIR filtering operations comes from the generalization of the "frequency-sampling method" [3]. The block diagram of the structure is given in Fig. 1. This structure is based on digital resonators embedded into a feedback loop. Due to this feedback, the properties of this structure substantially differ from that of the well-known Lagrange or frequency-sampling structures [3], [4].

In Section II coefficient sensitivity formulas are presented which promise nice sensitivity behavior at least in the vicinity of the resonator pole frequencies. In Section III the possible resonator pole positioning strategies are investigated, and a simple condition is derived, which guarantees the "passivity" of the feedback loop.

II. Derivation of the Sensitivity Formulas

In this paper we will concentrate on the time-invariant version of the common structure (see Fig. 1). For simplicity we do not consider the case of multiple resonator poles, thus we have in every channel of the structure as an internal transfer function

$$H_m(z)=\frac{g_m z^{-1}}{1-z_m z^{-1}}, \qquad m=0,1,\cdots,N-1 \quad (1)$$

Manuscript received August 10, 1987.
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IEEE Log Number 8822762.

0098-4094/88/0900-1195$01.00 ©1988 IEEE

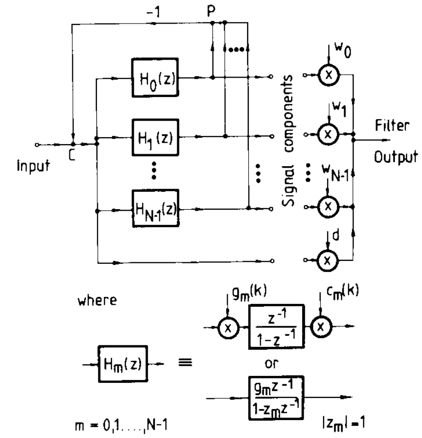

Fig. 1.

which is usually a function of complex coefficients having a pole on the unit circle. The global transfer function for every channel is

$$T_m(z) = \frac{H_m(z)}{1 + \sum_{n=0}^{N-1} H_n(z)}, \qquad m = 0, 1, \cdots, N-1 \tag{2}$$

while the overall transfer function of an FIR or IIR filter based on this structure can be written in the following form:

$$H(z) = d + \sum_{n=0}^{N-1} T_n(z)(w_n - d) = \frac{d + \sum_{n=0}^{N-1} w_n H_n(z)}{1 + \sum_{n=0}^{N-1} H_n(z)}. \tag{3}$$

In this paper we will suppose that the poles and zeros of $H(z)$ are real numbers or occur in complex-conjugate pairs, and as usual, even d is a real number.

It is a very interesting property of this structure that at the resonator pole positions

$$H(z_m) = w_m, \qquad m = 0, 1, \cdots, N-1 \tag{4}$$

from which we can deduce that the sensitivity of the transfer function at $z = z_m$ is zero with respect to any $H_n(z)$, w_n and z_n $(n = 0, 1, \cdots, N-1)$ except the $n = m$ case. This property is due to the infinite loop gain at these frequencies providing complete independency of the coefficients within the feedback loop.

It is very important to note that this common feedback implements a perfect pole–zero cancellation mechanism. The poles in (1), which in a general case are not necessarily located on the unit circle, will be transformed by the feedback loop into zeros, which automatically, and perfectly cancel their "generators" in expression (3). This is the reason why the application of ideal resonators does not cause implementational problems, since every resonator pole is cancelled by a zero generated by the overall common feedback from the very same resonator pole, and this is true even if implementational errors occur. The practical implementation of the resonators preferably involves those (first-, or second-order) structures that can locate (real, or complex-conjugate) resonator poles on the unit circle in such a way that coefficient quantization will affect only the "angle" of these poles.

The coefficient sensitivity formulas for the filters based on this common structure are as follows:

$$S_d^{H(z)} = \frac{d}{H(z)} \frac{\partial H(z)}{\partial d} = \left[1 - \sum_{n=0}^{N-1} T_n(z)\right] \frac{d}{H(z)} \tag{5}$$

$$S_{w_m}^{H(z)} = \frac{w_m}{H(z)} \frac{\partial H(z)}{\partial w_m} = T_m(z) \frac{w_m}{H(z)} \tag{6}$$

$$S_{g_m}^{H(z)} = \frac{g_m}{H(z)} \frac{\partial H(z)}{\partial g_m} = T_m(z) \left[\frac{w_m}{H(z)} - 1\right] \tag{7}$$

$$S_{z_m}^{H(z)} = \frac{z_m}{H(z)} \frac{\partial H(z)}{\partial z_m} = \frac{z_m z^{-1}}{1 - z_m z^{-1}} T_m(z) \left[\frac{w_m}{H(z)} - 1\right]$$
$$= \frac{z_m}{g_m} T_m^2(z) \left[1 - \frac{d}{w_m} + \sum_{\substack{n=0 \\ n \neq m}}^{N-1} \left(1 - \frac{w_n}{w_m}\right) H_n(z)\right] \frac{w_m}{H(z)}. \tag{8}$$

Since $H(z_m) = w_m$, $T_m(z_m) = 1$, and $T_m(z_n) = 0$ if $n \neq m$, from (5) and (7) it follows that in the vicinity of z_m the sensitivity with respect to d and g_m $(m = 0, 1, \cdots, N-1)$ will be small concerning both the magnitude and phase, and it will be zero if $z = z_n$, $n = 0, 1, \cdots, N-1$. The same is true for (6) and (8) except at z_m and its vicinity. For the weighting parameter w_m we have obviously $S_{w_m}^{H(z)} = 1$ at $z = z_m$, while for the "location sensitivity" we obtain from (8)

$$S_{z_m}^{H(z_m)} = \frac{z_m}{g_m} \left[1 - \frac{d}{w_m} + \sum_{\substack{n=0 \\ n \neq m}}^{N-1} \left(1 - \frac{w_n}{w_m}\right) H_n(z_m)\right],$$
$$m = 0, 1, \cdots, N-1. \tag{9}$$

It worth mentioning at this point that the magnitude sensitivity with respect to the pole "angle" $(z_m = e^{j\varphi_m})$ will also be zero at $z = z_m$ if the imaginary part of (9) is zero. If $w_n = 1$, $n = 0, 1, \cdots, N-1$, the condition of this zero magnitude sensitivity is rather simple:

$$g_m = z_m r_m, \qquad m = 0, 1, \cdots, N-1 \tag{10}$$

where r_m is a real constant. This condition also means that we will have zero magnitude sensitivity at the output of every channel of this structure to the pole angles, which is a nice property in transformer (or filter-bank) applications. In the case of the recursive Fourier transformation this condition is automatically fulfilled (see [1, expression (10)]).

III. Resonator Pole Positioning Strategies

The design of the filter parameters for this structure is rather straightforward. If the resonator positions are known, the weighting coefficients are given by (4), while if $\{p_n\}$, $n = 0, 1, \cdots, N-1$, are the poles of the filter

$$g_m = z_m \frac{\prod_{n=0}^{N-1} (1 - p_n z_m^{-1})}{\prod_{\substack{n=0 \\ n \neq m}}^{N-1} (1 - z_n z_m^{-1})} \tag{11}$$

see [4]. The resonator poles, at the price of some redundancy, can be located arbitrarily. Excellent passband behavior can be achieved if the resonator poles are distributed in the passband. For elliptic filters, to provide good stopband behavior and spare weighting coefficients, the resonator poles can be located to the positions of the zeros of the filter transfer function. Another strategy can optimize the dynamic range of the filter, etc.

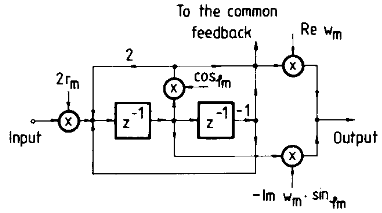

Fig. 2.

In the following we will show that the positioning strategy proposed by condition (10) (see also (11)) has very interesting consequences. First in Fig. 2 we present a simple second-order structure, which "structurally" forces the fulfilment of condition (10), and at the same time is canonic considering the number of delay elements and the number of nontrivial multiplications. Using this structure the resonator poles will be always on the unit circle, coefficient quantization will affect only the "angle" of these poles. It is important to note that this second-order section implements

$$G_m(z) = w_m H_m(z) + w_m^* H_m^*(z) \tag{12}$$

where asterisk denotes a complex conjugate.

As a next step let us investigate the behavior of the common feedback loop. The transfer function from the input to point P (see Fig. 1) has the following form:

$$H_P(z) = \frac{\sum_{n=0}^{N-1} H_n(z)}{1 + \sum_{n=0}^{N-1} H_n(z)}. \tag{13}$$

It is very easy to show that the magnitude of this transfer function is less or equal to the unity (i.e., it is "passive" in this sense), if

$$\mathrm{Re} \sum_{n=0}^{N-1} H_n(z) \geqslant -\frac{1}{2}. \tag{14}$$

Condition (14) can be fulfilled at every frequency if condition (10) holds, and

$$\sum_{m=0}^{N-1} \mathrm{Re}\left[\frac{g_m}{z_m}\right] = \sum_{m=0}^{N-1} r_m \leqslant 1. \tag{15}$$

The proof is straightforward, and will be omitted here. Equality condition in (15) is easily achieved if we have one more resonator pole than filter pole. This is due to the fact that the coefficient of z^{-N} in the denominator polynomial of $H(z)$ will be zero if in (15) equality holds. In this case $H_P(z)$ implements an all-pass filter, and if so, we know even its zeros, since they are in mirror image relationship with the poles of $H(z)$.

At this point of our development we can determine those resonator pole positions which will provide the above properties. These positions coincide with the zeros of $1 - H_P(z)$, since the input of the resonators (see point C in Fig. 1) is the difference of the filter input and the output at point P. We will have two sets of resonator poles, since the filter poles do not specify the sign of $H_P(z)$.

If the number of resonator poles equals the number of the filter poles, $H_P(z)$ cannot be an all-pass transfer function, because it is forced to have at least one zero at the origin, otherwise the loop would be delay free. The resonator pole positions, however, can be determined rather similarly by finding the zeros of $1 - H_A(z)$, where $H_A(z)$ is an all-pass function having the same poles as the filter has. These resonator poles will meet condition (10), and one of the two resonator pole sets will insure also the fulfillment of condition (15).

In the case of the recursive Fourier transformation $r_m = 1/N$ $(m = 0, 1, \cdots, N-1)$, thus condition (15) is fulfilled, and $H_P(z) = z^{-N}$, that not surprisingly will provide as resonator poles the Nth roots of the unity. The companion set consists of the Nth roots of -1, and generates a "companion" recursive Fourier transformation that can play some role, if the zero frequency component of the signal should be suppressed.

IV. Conclusions

In this paper the sensitivity properties of a recently introduced resonator-based structure has been presented. This structure, from some respects of the filter design, is closely related to the frequency-sampling structure, however, due to a common feedback, has low sensitivity to the coefficients. The common feedback provided perfect pole–zero cancellation, thus the application of ideal resonators does not cause implementational problems. It is also shown that the resonator poles can be arbitrarily located on the unit circle, however, there exists a strategy which results in a canonic solution, and insures the "passivity" of the common feedback loop.

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Bilinear Form Approach to Synthesis of a Class of Electric Circuit Digital Signal Processing Algorithms

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***Abstract* —A number of different digital signal processing algorithms for Electric Power System data acquisition, control and protection were introduced in the past. Each of the algorithms was defined based on the specific application utilizing either some heuristic approaches or known systems identification and parameter estimation techniques. A generalized methodology for algorithm synthesis which may be used in a number of different applications is proposed in this paper based on the Bilinear Form approach.**

Manuscript received December 8, 1987.

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IEEE Log Number 8822766.

0098-4094/88/0900-1197$01.00