Digital Filters Based on Recursive Walsh–Hadamard Transformation

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Abstract—In this paper the applicability of the recursive Walsh–Hadamard transformation [1] to FIR and IIR filtering is investigated. It is shown that using the recently introduced common structure for recursive transforms [2] the usual frequency-domain FIR filtering problem can be easily converted into a Walsh sequency-domain filtering problem. It is also shown that a simple modification of this structure results in a possible alternative for IIR filter implementations.

I. INTRODUCTION

It is well known from the literature that the Walsh–Hadamard transformation (WHT) [3], and especially its fast version, can be efficiently used for the calculation of the discrete Fourier transform (DFT) [4], for implementing adaptive filters [5], and for DFT spectrum filter realizations [6]. In this paper, the recursive form of the WHT is investigated, and for its implementation the recently introduced common structure for recursive transformations [2] is suggested.

The derivation of this common structure is based on the state variable formulation and the results of the observer theory [1], while its applicability to FIR and IIR filtering operations comes from the generalization of the "frequency sampling method" [7]. The block diagram of the proposed structure is given in Fig. 1. For every discrete transformation to provide a dead-beat observer behavior, the \( c_m(k) \) and \( g_m(k) \), \( m = 0,1,\ldots,N-1 \), values should be the \( k \)th components of the basis and reciprocal basis vectors of the transformation, respectively [2].

In this paper we concentrate on the recursive Walsh–Hadamard transformation [1], which is obtained if in Fig. 1:

\[
\begin{align*}
    c_m(k) &= \text{wald}(m,k) \\
    g_m(k) &= \frac{1}{N} \text{wald}(m,k)
\end{align*}
\]

and as usual, \( N \) is an integer power of 2. This transformation is of practical interest, since multiplication by (1) can be reduced to additions, and a single division by \( N \), that is a simple shift operation.

In Section II the properties of the recursive WHT structure are examined. It is shown what is the relation between the recursive DFT and WHT, and what is the condition of "structural passivity" [8] for the recursive WHT. Section III presents a new method for implementing IIR filters. This method is based on a modified form of the recursive WHT structure. In Section IV some aspects of the hardware realization are discussed.

II. PROPERTIES OF THE RECURSIVE WALSH–HADAMARD TRANSFORMER

Since the overall behavior of the transformer in Fig. 1 is time invariant, for the analysis we can replace the internally time-invariant components, in every channel, by the transfer function \( K_m(z) \), \( m = 0,1,\ldots,N-1 \), as it is shown in Fig. 2. \( K_m(z) \) is a function of real coefficients, and can be decomposed as a linear combination of elementary transfer functions having one (complex) pole located on the unit circle [9]. These resonator-pole positions are the \( N \)th roots of the unity \((z_n, n = 0,1,\ldots,N-1)\), and the "weight" of these elementary transfer functions comes from the Fourier series expansion of the Walsh–Hadamard functions applied. If the input signal of the transformer is band-limited in the usual sense, i.e., by one half of the sampling frequency, the effect of "modulating" by the discrete Walsh functions is equivalent to the effect of a limited number of discrete harmonical components. Thus

\[
K_m(z) = \sum_{n=0}^{N-1} \nu_n H_n(z)
\]

where \( \{ \nu_n \}, \ m = 0,1,\ldots,N-1 \), are the appropriate weighting coefficients, and

\[
H_n(z) = \frac{1}{N} \frac{z_n z^{-1}}{1 - z_n z^{-1}}, \quad n = 0,1,\ldots,N-1
\]

is the elementary transfer function. It is interesting to note that for the recursive DFT \( K_m(z) = H_m(z), m = 0,1,\ldots,N-1 \) [2].

Concerning the stability of the recursive structure (see Fig. 2) the transfer function from the input to point \( P \) should be investigated. The stability of this feedback loop in the linear sense comes from the fact that all the poles of the transfer function

\[
H_p(z) = \sum_{n=0}^{N-1} K_n(z)
\]

are at the origin of the \( z \)-plane unless coefficient quantization errors occur. Note, however, that for the recursive Walsh–Hadamard transformer the only source of coefficient quantization error is the division by \( N \), and if we apply "magnitude-truncation" strategy, it is easy to show, that both the frequency domain criteria given in [10], and the structural boundedness property detailed in [8] can be fulfilled. This later property is simply due to the fact that \( H_p(z) \) in (4) implements an all-pass transfer function, the magnitude of which can not exceed the unity if "magnitude-truncation" is applied. Thus in the Walsh–Hadamard transformer implemented as in Fig. 1, the zero-input limit cycles can be suppressed (see also [11]).

The implementation of FIR filters using this transformer is rather straightforward, since only the appropriate linear combination of the transformer outputs should be generated. These

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Fig. 1. The suggested common structure.

Fig. 2. "Time-invariant" Walsh–Hadamard transformer.

Fig. 3. Modified Walsh–Hadamard transformer.

"Taps" do not influence the stability of the transformer structure. Thus the transfer function for an FIR filter has the form of

\[
H(z) = \frac{d' + \sum_{n=0}^{N-1} w'_n K_n(z)}{1 + \sum_{n=0}^{N-1} K_n(z)}
\]

where the \(d'\) and \(w'_n, n = 0, 1, \ldots, N-1\), are the weighting coefficients. The calculation of these coefficients is relatively simple. By introducing \(\{v_{m}\} = V', w = [w_0, w_1, \ldots, w_{N-1}]^T, w' = [w'_0, w'_1, \ldots, w'_{N-1}]^T\), where

\[
w_m = H(z_m), \quad m = 0, 1, \ldots, N-1
\]
as for the "frequency sampling method" [7], we have

\[
w' = V^{-T}w
\]

(here \(V^{-T}\) stands for "transpose of the inverse of \(V\)"). and finally,

\[
d' = H(0).
\]

The calculation of \(V\) is straightforward

\[
V = \{v_{m}\} = WF^{-1}
\]

where \(W\) is the Walsh–Hadamard, while \(F\) is the discrete Fourier transformation matrix.

\section{III. THE IMPLEMENTATION OF IIR FILTERS}

If we introduce additional weighting coefficients into the transformer structure like in Fig. 3, the implementation of IIR filters is also possible. For the recursive Fourier transformation these coefficients are given by

\[
\gamma_m = \prod_{n=0}^{N-1} \left(1 - p_n z_m^{-1}\right), \quad m = 0, 1, \ldots, N-1
\]

where \(\{p_n\}, n = 0, 1, \ldots, N-1\), are the poles of the filter (see [2]). The number \(\gamma_m\) is usually complex, but if the poles are real numbers, or occur in complex-conjugate pairs, the complex conjugate of \(\gamma_m\) will also appear.

Let us denote \(\gamma = [\gamma_0, \gamma_1, \ldots, \gamma_{N-1}]^T, \gamma' = [\gamma'_0, \gamma'_1, \ldots, \gamma'_{N-1}]^T\), and apply the structure of Fig. 3, then we have for the IIR filter

\[
gamma' = V^{-T}g.
\]

The tap coefficients can be derived from (7) if \(w'_n\) and \(w''_n\) are replaced by \(w_{m}^*\gamma_m\) and \(w_{m}^*\gamma_m^*\), respectively, \((m = 0, 1, \ldots, N-1)\). It is interesting to note that if the transfer function to be implemented is of real coefficients, then both the \(w_{m}\) and \(\gamma_m\) values \((m = 0, 1, \ldots, N-1)\) will be real numbers.

It is worth mentioning, however, that the introduction of the \(\gamma_m\) parameters destroys the orthogonality of the transformer structure, and thus the general criteria of nonlinear stability and low sensitivity [8] are not fulfilled. Still for its computational simplicity in certain cases even this Walsh–Hadamard transformer can be suitable for IIR filter realizations.
IV. IMPLEMENTATIONAL ASPECTS

The hardware implementation of the recursive Walsh-Hadamard transformer is relatively simple, since it requires only additions and a single shift operation to perform division by $N$. The Walsh function samples can be stored in a memory, or can be generated by a simple digital circuit. To implement FIR and IIR filters the multiplications by $d^i, w_m^i$, and $y_m^i, m = 0, 1, \ldots, N - 1$, obviously can not be avoided, however, this structure is still canonic with respect to the number of (nontrivial) multiplications.

The recursive Walsh-Hadamard transformer has been implemented by the authors on a microprogrammable signal processor based on bit-sliced elements [12]. It was found that for this hardware, and for every input sample, the number of microprogram steps to perform Walsh-Hadamard transformation is

$$S(N) = 6N + \log_2 N + 13,$$

where $N$ equals the transform size. The number $S(N)$ includes the steps required for the Walsh function generation as well.

V. CONCLUSIONS

In this paper the applicability of the recursive Walsh-Hadamard transformer has been investigated. It turned out that using the common structure proposed, both FIR and IIR filters can be efficiently implemented, and that for the transformer itself, and for FIR filters, the structural boundedness of the common feedback loop can be easily guaranteed. This property implies the ability of suppressing zero-input limit cycles. The generalization of these results for arbitrary discrete transformation is straightforward.

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