turn, this choice led to an approximately unity eigenvalue spread for the covariance matrix of the reference signal.

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Convergence Properties of an Adaptive Fourier Analyzer

Gyula Simon and Gábor Péceli

***Abstract*— A PLL-like adaptive Fourier analyzer (AFA) was proposed which has shown excellent performance in practical applications. The convergence analysis of this AFA is extremely difficult, and until now theoretical results have not been available. In this paper a modified version of the original AFA will be proposed. The new version preserves the effectiveness of the original AFA, and its convergence properties can be exactly analyzed. Sufficient conditions are presented for the exponential stability, and the absolutely monotone convergence, as a function of the harmonic content of the input signal. The speed of the convergence is also estimated, and the effect of the noise and of unmodeled periodic components are analyzed.**

***Index Terms*— Adaptive filters, asymptotic stability, observers, resonators, spectral analysis, stability criteria.**

I. INTRODUCTION

The resonator bank structure is an attractive tool for implementing transformations [2]. Based on this structure, new adaptive algorithms were proposed to track the input frequency (or frequencies), and to provide accurate harmonic component measurement as well. In [3] a hyperstable structure was proposed to track independent sinusoidal signal components, but the calculation of the Fourier coefficients and the frequencies is difficult. In many practical cases, hyperstability is an unnecessarily strict requirement, but the robustness of the algorithm in a well-defined region is essential. A different method proposed in [1] is capable of tracking periodic signals, and of calculating the fundamental frequency as well as the Fourier coefficients in real time. There is excellent practical experience concerning the robustness of the method, but no stability analysis is available as yet. In this paper a somewhat different version of the AFA, the so-called block AFA (BAFA), is proposed. According to experiments, the new algorithm preserves the good performance properties of the original version.

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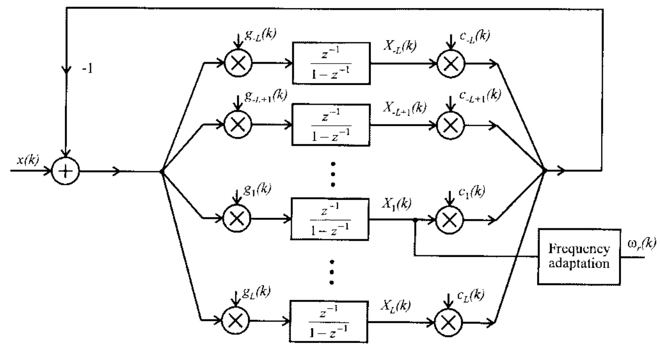


Fig. 1. The block diagram of the BAFA.

The convergence analysis is carried out from a worst-case viewpoint providing sufficient conditions for exponential stability and absolutely monotone convergence. This approach also allows the estimation of the convergence speed. All the presented results depend strongly on the input signal, of which some *a piori* knowledge is necessary.

In Section II, the BAFA structure is introduced. Section III contains the results of the convergence analysis, the sufficient conditions for the exponential stability, and the absolutely monotone convergence. A lower bound on the convergence speed is also derived, and the effect of the noise and unmodeled periodic components are analyzed.

In Section IV some examples are presented comparing the convergence of the AFA and the BAFA, and illustrating the convergence analysis.

II. THE BLOCK AFA

The signal processing part of the system is a special single input multiple output (SIMO) filter bank, which can be considered as an observer [4]. The system to be observed consists of a set of harmonically related resonators. In Fig. 1, these resonators are composed of complex-valued discrete integrators, which estimate the corresponding Fourier coefficients X_i, and of complex modulators, which generate the Fourier components. The sum of the components is compared to the input sample in every step, and if there is a difference, all the coefficient estimates will be corrected. The system provides the Fourier coefficients/components recursively if, in Fig. 1,

$$\left.\begin{aligned} c_m(k) &= e^{j\omega_r mk} \\ g_m(k) &= r_m e^{-j\omega_r mk} \end{aligned}\right\} \qquad m = -L, -L+1, \cdots, 0, \cdots, L \tag{1}$$

where ω_r is the fundamental resonator frequency and $N = 2L+1$ is the number of (complex) resonators. If the coefficients r_m are set as

$$r_m = \frac{1}{\prod_{\substack{i=-L \\ i\neq m}}^{L} (1 - z_i z_m^{-1})}, \qquad z_i = e^{j\omega_r i} \tag{2}$$

the system has poles only at the origin, therefore, it will operate as a deadbeat observer with transient length of N.

The input signal $x(k)$ is periodic:

$$x(k) = \sum_{i=-K}^{K} A_i e^{j\omega_0 ik} \tag{3}$$

where ω_0 is the fundamental frequency, A_i is the complex amplitude of the ith component, and the number of complex harmonics is $2K+1$ (it is supposed that $K\omega_0$ is smaller than the Nyquist frequency).

If $\omega_0 = \omega_r$, then the coefficient estimators X_i equal the exact Fourier coefficients A_i. When the frequency of the input differs from that of the model, the estimated Fourier coefficients suffer from the effects of leakage and picket fence. If the frequency estimate is ω_r, the state variable X_1 corresponding to the fundamental harmonic has the following form:

$$\begin{aligned} X_1(k) &= \sum_{i=-K}^{K} X_{1,i}(k) = \sum_{i=-K}^{K} A_i e^{j(\omega_0 i - \omega_r)k} T_1(e^{j\omega_0 i}) \\ &= \sum_{i=-K}^{K} A_i' e^{j(\omega_0 i - \omega_r)k} \end{aligned} \tag{4}$$

where the transfer function of the mth channel from $x(k)$ to $X_m(k)c_m(k)$ is

$$T_m(z) = \frac{\dfrac{r_m z_m z^{-1}}{1 - z_m z^{-1}}}{1 + \sum_{i=-L}^{L} \dfrac{r_i z_i z^{-1}}{1 - z_i z^{-1}}}, \qquad z_i = e^{j\omega_r i}. \tag{5}$$

It is clear that angle $X_{1,1}(k+1)$ − angle $X_{1,1}(k) = \omega_0 - \omega_r$. If $A_i' \ll A_1', i \neq 1$, then X_1 is a good estimate of $X_{1,1}$, and the frequency error can be calculated from the angle of the state variable:

$$\Delta\varphi = \text{angle } X_1(k+P) - \text{angle } X_1(k), \quad \omega_0 - \omega_r \approx \Delta\varphi/P \tag{6}$$

where P is a positive integer.

Based on these results, the adaptation mechanism of the BAFA can be described as follows.

1) Wait N samples until the transients of the resonator bank decay.
2) Collect P samples of X_1, and measure $\Delta\varphi$ as in (6). Note that if the unwrapped angle difference between two samples of X_1 is higher than π, then the exact angle measurement is not possible!
3) Update ω_r:

$$\omega_{r,\text{new}} = \omega_{r,\text{old}} + \Delta\varphi/P. \tag{7}$$

Update L:

$$\pi/\omega_{r,\text{new}} - 1 \leq L < \pi/\omega_{r,\text{new}} \text{ where } L \text{ is an integer.} \tag{8}$$

Update r_m as in (2).
4) Go to i.

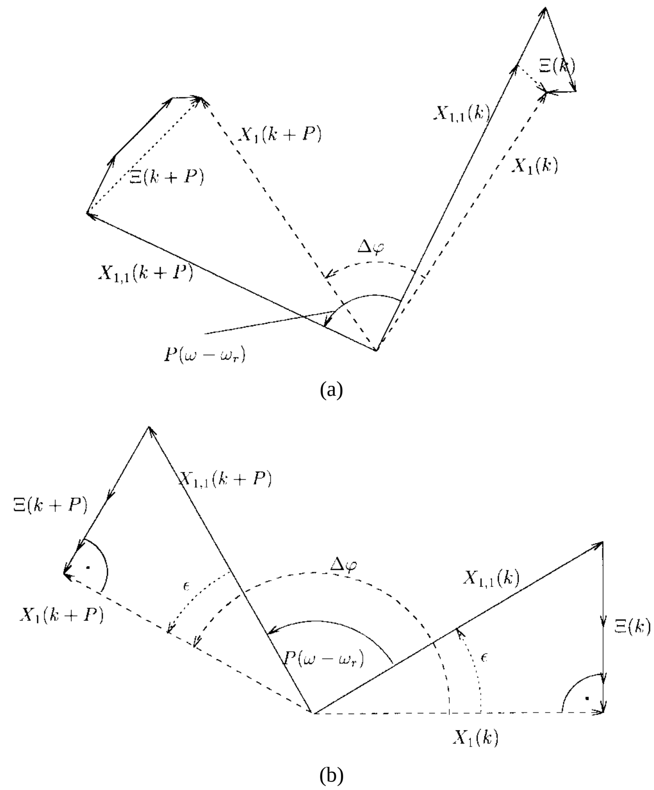


Fig. 2. (a) The effect of "disturbing" harmonic components on the angle measurement and (b) worst case.

III. Convergence Analysis

The following model is used for the input signal:

- The signal is periodic with unknown fundamental frequency ω_0 as in (3).
- There exist *a priori* upper limits for the relative amplitudes of the harmonic components:

$$a_i \geq \frac{|A_i|}{|A_1|}. \tag{9}$$

Let Ξ be the sum of the "disturbing" components of X_1:

$$\Xi(k) = \sum_{i=-K,\, i\neq 1}^{K} X_{1,i}(k). \tag{10}$$

If Ξ is not zero, then $\Delta\varphi \neq P(\omega_0 - \omega_r)$, as shown in Fig. 2(a). The angle error is the largest, when $|\Xi|$ is the largest, i.e., the components of X_1 have the same angle, and Ξ is perpendicular to X_1 as shown in Fig. 2(b). Using the notations of Fig. 2(b), the angle error

$$\begin{aligned} &|P(\omega_0 - \omega_r) - \Delta\varphi| \\ &\quad \leq 2\varepsilon = 2\arcsin\frac{|\Xi|}{|X_{1,1}|} \\ &\quad \leq 2\arcsin\frac{\sum_{\substack{i=-K \\ i\neq 1}}^{K} |T_1(e^{j\omega_0 i})A_i|}{|T_1(e^{j\omega_0})A_1|} \leq 2\arcsin\frac{\Gamma_1}{\Lambda_1} \end{aligned} \tag{11}$$

where

$$\Gamma_m = \sum_{\substack{i=-K \\ i\neq m}}^{K} a_i |T_m(e^{j\omega_0 i})| \quad \text{and} \quad \Lambda_m = |T_m(e^{j\omega_0 m})|. \tag{12}$$

Theorem: If the actual frequency estimate is ω_r, and the update mechanism is as in 3) above, then for the absolute frequency error to decrease in the next adaptation step, it is sufficient that

$$\Gamma_1 < \Lambda_1 \tag{13}$$

and

$$|\omega_0 - \omega_r| + 2\arcsin\frac{\Gamma_1}{\Lambda_1} < \pi \tag{14}$$

and

$$\frac{2}{P}\arcsin\frac{\Gamma_1}{\Lambda_1} < |\omega_0 - \omega_r|. \tag{15}$$

Proof: Conditions (13) and (14) simply ensure that the angle difference remains below π between two samples, which is necessary for correct angle measurement. Using (7), (11), and (15), the frequency error can be expressed as follows:

$$\begin{aligned} |\omega_0 - \omega_{r,\text{new}}| &= \left|\omega_0 - \omega_{r,\text{old}} - \frac{\Delta\varphi}{P}\right| \\ &= \frac{1}{P}\,|P(\omega_0 - \omega_{r,\text{old}}) - \Delta\varphi| < |\omega_0 - \omega_{r,\text{old}}| \end{aligned} \tag{16}$$

which guarantees the decrease of the absolute frequency error after the adaptation step. ■

Note that if (13) and (14) hold, there always exists a sufficiently large P to fulfill (15).

It is clear, that if the theorem holds in a domain $(\omega_{0,\min} < \omega_0 < \omega_{0,\max},\ \omega_{r,\min} < \omega_r < \omega_{r,\max})$, then starting from a point inside of the domain the BAFA will be absolutely monotone convergent.

If for a $0 < \delta < 1$ value it is true that in the above domain the ratio of the left-hand side and right-hand side of (15) is smaller than δ, then it can be shown similarly to (16) that the frequency error will be at least δ times smaller in each adaptation step:

$$|\omega_0 - \omega_{r,\text{new}}| < \delta|\omega_0 - \omega_{r,\text{new}}|. \tag{17}$$

The property (17) assures that the frequency estimator ω_r is exponentially stable (and thus uniformly asymptotically stable) [5], and also gives an estimation on the speed of the convergence.

Because of the exponential stability, the frequency estimator can be arbitrarily close to the true frequency value after a certain time instant, therefore, the errors of the amplitude estimators, coming from the leakage and the picket-fence effect, can also be arbitrarily small after this time instant (independently of the initial state), so the estimators of the Fourier coefficients are uniformly asymptotically stable [5].

In the presence of disturbances (either noise or unmodeled periodic components), the frequency and amplitude estimators are perturbed. But the uniformly asymptotic stability yields that the estimators are totally stable [5], which means that in the presence of disturbances, the estimators can be arbitrarily close to the true values, if the disturbance is sufficiently small, and the initial states are close enough to the solution. The total stability property assures that the algorithm is not "too sensitive" to the disturbances. A more exact convergence analysis can be carried out based on the method proposed in the noise-free case. Both periodic and stochastic disturbances will be analyzed.

Let the sum of the unmodeled periodic disturbances be

$$P_d(k) = \sum_{i=1}^{Q} B_i e^{j\omega_{d,i}k} \tag{18}$$

where Q is the number of components, and B_i is the amplitude of the complex periodic signal with frequency $\omega_{d,i}$. The maximum amplitude error caused by P_d on the X_m output can be expressed as

$$\gamma_m = \sum_{i=1}^{Q} B_i|T_m(e^{j\omega_{d,i}})|. \tag{19}$$

Since the resonator bank operates in steady-state mode during step 2) above, the effect of the disturbing periodic components on the X_1 variable is additive, and is similar to that of the signal components $X_{1,i}, i \neq 1$ in (10). The analysis leads again to (13)–(15), where the variable Γ_m now has the form

$$\Gamma_m = \sum_{\substack{i=-K \\ i\neq m}}^{K} a_i|T_m(e^{j\omega_0 i})| + \frac{\gamma_m}{A_1}. \tag{20}$$

If the disturbance can be modeled as an additive stationary $n(k)$ noise with variance σ^2, and power spectral density function $S(\omega)$, then the output noise $n_m(k)$ on the Fourier coefficient X_m can be characterized as a stochastic variable with real and imaginary parts having zero mean Gaussian distribution [6]. The variance σ_m^2 of $n_m(k)$ can be expressed as

$$\sigma_m^2 = \int_{-\pi}^{\pi} S(\omega)|T_m(e^{j\omega})|^2\,d\omega. \tag{21}$$

It is obvious that the variances of the real and imaginary parts of $n_m(k)$ cannot be larger than σ_m^2. Thus, with a confidence level of 99.7%, it is true that

$$|\mathrm{Re}\,n_m(k)| < 3\sigma_m$$

and

$$|\mathrm{Im}\,n_m(k)| < 3\sigma_m \tag{22}$$

which yields

$$|n_m(k)| < 3\sqrt{2}\sigma_m. \tag{23}$$

In the special case when $n(k)$ is white, the probability density function of $n_m(k)$ is a circularly symmetric Gaussian distribution with $\sigma_{m,\mathrm{Re}}^2 = \sigma_{m,\mathrm{Im}}^2 = \sigma_m^2/2$, and (21) gives $\sigma_m^2 \cong \sigma^2/N$ (the equation holds exactly if $\omega_r = 2\pi/N$, otherwise for high N it is a good estimation). The stochastic variable $|n_m(k)|^2$ has chi-square distribution with degree of 2, which can be expressed as an exponential distribution with coefficient 1/2. For confidence level α

$$P\left(|n_m(k)|^2 < \frac{\sigma_m^2}{2}\,x\right) = 1 - e^{-\pi/2} = \alpha \tag{24}$$

which gives $x = -2\ln(1-\alpha)$, and the α confidence level upper limit for the output noise is

$$|n_m(k)| < \frac{\sigma_m}{\sqrt{2}}\sqrt{-2\ln(1-\alpha)} \approx \sigma\sqrt{-\ln(1-\alpha)/N}. \tag{25}$$

The disturbing noise component n_1 can be added to the variable Ξ. If $\eta_m > n_m(k)$ for all k, then the sufficient conditions for the absolute frequency error to decrease in the next adaptation step are again (13)–(15) with

$$\Gamma_m = \sum_{\substack{i=-K \\ i\neq m}}^{K} a_i|T_m(e^{j\omega_0 i})| + \frac{\eta_m}{A_1}. \tag{26}$$

If both periodic and stochastic disturbances are present, from (20) and (26), Γ has the following form:

$$\Gamma_m = \sum_{\substack{i=-K \\ i\neq m}}^{K} a_i|T_m(e^{j\omega_0 i})| + \frac{\gamma_m + \eta_m}{A_1}. \tag{27}$$

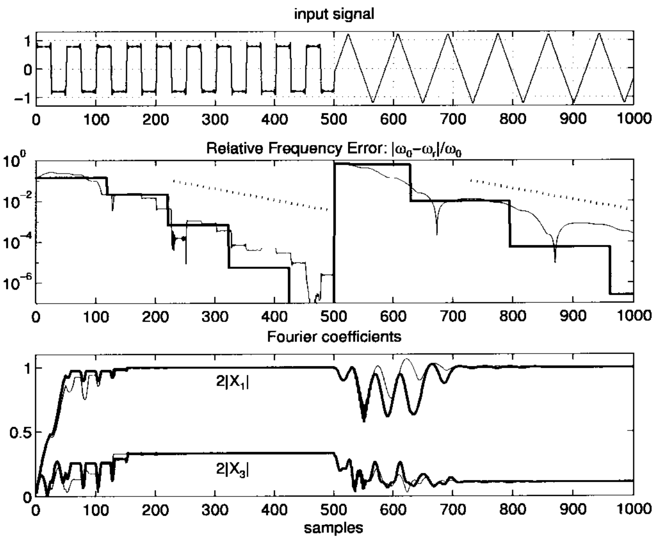


Fig. 3. Convergence of the frequency estimator, and two amplitude estimators of the AFA (thin line), and the BAFA (thick line). Dotted lines show worst case estimations of the convergence rate of ω_r.

The disturbances have the following two effects on the convergence.

- Conditions (13) and (14) are not fulfilled close to the edge of the noise-free convergence interval, thus the convergence interval becomes narrower.
- Condition (15) cannot be fulfilled when $\omega_r - \omega_0$ is close to zero, thus the convergence interval is "holed," there is a residual $\Delta\omega$ frequency error which can be estimated using (15) as

$$\Delta\omega = |\omega_0 - \omega_r| \approx \frac{2}{P} \arcsin \frac{\Gamma_1}{\Lambda_1}. \tag{28}$$

For small disturbances, $\Delta\omega$ is small, thus $\Lambda_1 \approx 1$, the first term of Γ_1 in (27) is close to zero, and (28) can be approximated by

$$\Delta\omega \approx 2\, \frac{\gamma_1 + \eta_1}{PA_1}. \tag{29}$$

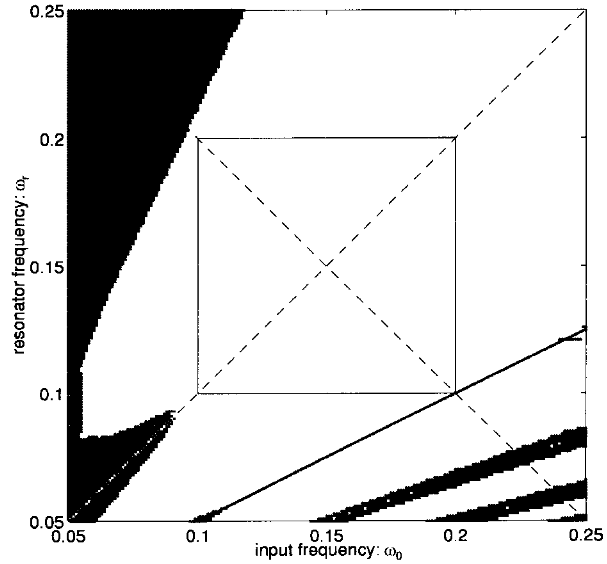
The error of the estimated Fourier coefficients comes directly from the disturbances and indirectly from the inexact frequency estimation. The direct error $\Delta A_{m,\,dir}$ is maximum $\gamma_m + \eta_m$ for the mth harmonic component. The indirect error is caused by the well-known picket-fence effect and the leakage, hence the amplitude estimators are not computed on the exact frequency lines. The $\Delta A_{m,\,ind}$ amplitude error on channel m caused by the picket fence effect and leakage can be estimated using (29):

$$\Delta A_{m,\,ind} < \max_{\omega_0 - \Delta\omega < \omega_r < \omega_0 + \Delta\omega} \left(A_m |1 - |T_m(e^{j\omega_0 m})|| + \sum_{\substack{i=-K \\ i \neq m}}^{K} A_i |T_m(e^{j\omega_0 i})| \right). \tag{30}$$

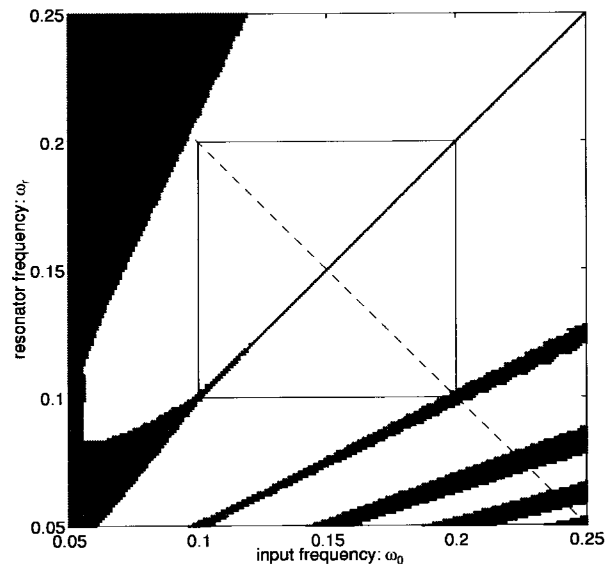
For the amplitude error ΔA_m, the following estimation can be given by summing $\Delta A_{m,\,dir}$ and $\Delta A_{m,\,ind}$:

$$\Delta A_m < \max_{\omega_0 - \Delta\omega\, <\, \omega_r\, <\, \omega_0 + \Delta\omega} (A_m |1 - \Lambda_m| + A_1 \Gamma_m) \tag{31}$$

where Γ has the form of (27).



(a)



(b)

Fig. 4. An example of the convergence analysis. (a) Noise-free input. Starting from a white point the absolute frequency error decreases in the next adaptation step. The rectangle indicates a region where the BAFA is exponentially stable. (b) In the presence of noise (SNR = 37 dB), the stability region becomes slightly smaller; and around $\omega_r = \omega_0$, the narrow gap of the residual error is visible.

IV. Examples

Fig. 3 compares the convergence of the AFA and the BAFA through an example. The input signal changed its shape (from square to triangle) and frequency (from $\omega_{0,1} = 0.125$ to $\omega_{0,2} = 0.075$) in step 500, as shown in the figure. The signal was filtered to fulfill the Nyquist criterion, and P was set to N in each adaptation step. Fig. 3 shows the convergence of the frequency estimator and the amplitude estimators of frequency lines 1 and 3. The performances of the algorithms are close to each other (which is typical, according to our experience, independently of the signal's properties), but the frequency estimator of the BAFA converges monotonously. A worst-case estimation of the convergence rate can also be seen on the plots based on (17), where δ was calculated from (15).

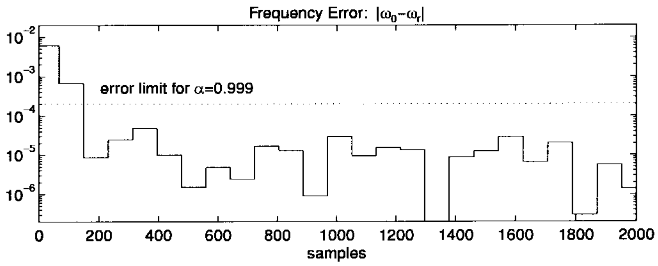


Fig. 5. Convergence and residual error of the frequency estimator in the presence of noise (SNR = 37 dB).

The second example is a convergence analysis of a signal with $A_1 = 1$, and $A_i < 1/i^2$ for the other harmonics. The frequency of the input signal is known to vary around 0.15. The goal is to determine the region of the input frequency in which the absolute monotone convergence can be ensured when $P < 40$. Equations (13)–(15) were evaluated by a computer program in a grid containing 200×200 points. On the plot, dark dots show the places where any one of the constraints does not hold. Fig. 4(a) shows the result of the analysis in the noise-free case. It is clear from the figure that for an input frequency $0.1 < \omega_0 < 0.2$, the absolutely monotone convergence can be ensured, provided ω_r is forced to start from the same interval. Fig. 4(b) shows the convergence analysis of the same signal corrupted by additive white noise with $\sigma \leq 0.01$ $(\mathrm{SNR} > 37$ dB). Now the convergence interval is slightly narrower, and in the middle of the convergence interval, a narrow "hole," the band of the residual frequency error, is visible. (For unmodeled periodic signals, a similar result can be obtained.) Using (25), (26), and (29), the frequency error is $\Delta\omega = 2 \cdot 10^{-4}$ with a confidence level of $\alpha = 99.9\%$. Fig. 5 shows the convergence curve of the frequency estimator of a triangle wave with $\omega_0 = 0.15$ and SNR = 37 dB. In the record, the residual error is smaller than $5 \cdot 10^{-5}$, which corresponds well with the theoretical limit of $\Delta\omega$.

V. Conclusions

The results presented in this paper allow the design of an adaptive algorithm, which is exponentially stable and absolutely monotone convergent in a region which depends on the harmonic content of the input signal. A lower bound on the convergence speed was given, which is important in real time applications. The effect of the periodic and stochastic disturbances was also analyzed, and the error of the frequency and amplitude estimators was also derived. Numerical examples were presented to illustrate the theoretical results.

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