IEEE Instrumentation and Measurement
Technology Conference
Brussels, Belgium, June 4-6, 1996

Generalization of the Frequency Sampling Method[1]

Gábor Péceli, Gyula Simon

Department of Measurement and Instrument Engineering,
Technical University of Budapest, H-1521 Budapest, Hungary
Phone: + 36 1 4632057 Fax: + 36 1 4634112 E-Mail: peceli@mmt.bme.hu

Abstract—**The classical "frequency sampling method" [1] based on the direct utilization of the Lagrange interpolation technique is extended in a natural way to a rather efficient Hermite interpolation scheme. This structure is a new extension related to the recently introduced resonator-based digital filter family [7]. The development resulted in a system having good properties if measurement of signals consisting of sinusoidal components is to be performed.**

I. INTRODUCTION

The "frequency sampling (FS) method" is a classical approach to design finite impulse response (FIR) digital filters (see e.g. [1]) in the frequency domain via the direct utilization of the Lagrange interpolation technique (see e.g. [2]). The FS method is strongly related to the recursive evaluation of the Discrete Fourier Transformation (DFT) if the frequency-domain sampling is uniform. Some early attempts to realize the recursive DFT suffered from serious implementational problems due to the fact that digital resonators operating at their stability limit were involved. Later this problem has been solved with the introduction of a common structure for recursive transformations [3] which proved to be useful in implementing different digital filters and filter-banks including infinite impulse response (IIR) and adaptive filters, as well ([4]-[7]). This structure consists of parallel first- or second-order digital resonators within a common feedback loop. The infinite loop gain at the resonator frequencies assures good sensitivity properties and the conditions of limit cycle immunity and low roundoff noise can also be fulfilled [4]. The transfer value from the filter input to one of the resonator outputs can be characterized by 1 at the corresponding resonator pole frequencies and by 0 at the pole frequency of all the other parallel resonator sections. The concept of "frequency sampling" means the prescription of transfer function values at these distinct, not necessarily uniformly distributed resonator pole positions. Typical frequency domain filter designs start with a tolerance scheme given as a desired magnitude, phase, or possibly group delay requirement. With the FS method the situation is somewhat different because it solves the approximation problem by interpolation and therefore the specifications are not automatically met. On the other hand at the resonance frequencies the prescribed values are implemented without any systematic error. This feature of the FS method is very attractive in solving measurement problems where the overall accuracy of the system is of key importance. A typical example can be the measurement of composite sinusoidal waveforms (see e.g. [8]) where there is a real chance to prescribe the necessary transfer values for all the frequency components to be measured.

Unfortunately, however, possibly with the exception of the synchronized multi-sine measurements, where the artificially generated sine waves are to be measured, the practical signals usually have considerable frequency content also between the "sampled" frequency positions. Therefore it isn't indifferent what is the behavior of the frequency characteristics in these ranges. The idea to generalize the FS methods comes at this point. On one hand it is desirable to maintain all the positive features of the original technique but on the other some additional approximation power is needed to improve the overall performance. In this paper the case of resonators with higher (> 1) multiplicity is investigated. This means that we have a structure consisting of cascades of identical resonators in parallel within a common feedback loop. The output taps of the multiple resonators may fix not only the complex transfer value but according to the actual resonator multiplicity its first, second, etc. derivatives at the corresponding frequency. It is easy to show that this technique is in complete correspondence with the Hermite interpolation method, however, similarly to the Lagrange interpolation case (see e.g. [4] or [7]) the very same structure is capable to combine the zero-set coming from the Hermite interpolation with an arbitrary pole-set and thus implement any rational transfer function.

The first part of the paper overviews all those specialties which come from measurement problems and strongly influence the design of digital filters and filter-banks. Zero

---

[1] This work was supported by the Hungarian Fund for Scientific Research (OTKA) under contract 4-100.

0-7803-3312-8/96/$5.00©1996 IEEE

group delay and consequently constant phase in a certain frequency range is a typical requirement e.g. in measuring bridges. The second part is devoted to the analysis of the structure consisting of parallel resonators and to the case where higher-order resonators are applied. In this section the design equations are also provided. Finally in the third part the applicability of this Hermite interpolation structure to adaptive Fourier analysis (see e.g. [6]) is investigated.

II. PARALLEL MEASUREMENT OF SINUSOIDAL SIGNAL COMPONENTS

There are many practical problems where the measurement of signals containing sinusoidal components is required. Typical examples are the measurements related to rotating machinery or simply to the line frequency. In both cases the harmonically related sinusoidal components are to be separated and the unavoidable measurement noise is to be suppressed. The recursive DFT structure suggested in [3] can be a good candidate to perform measurements on such signals if the frequency of the components is known in advance or can be determined by an adaptation mechanism [6]. The block diagram of such a transformer is given in Fig. 1. The structure consist of complex coefficient first-order resonators having a single complex pole on the unit circle. All these resonators operate in parallel within a common feedback loop. At the frequencies corresponding to the resonator pole frequencies this loop has infinite loop gain therefore the transfer value equals to 1 independently of the other parameters of the system. The transfer function of the mth channel of this filter-bank has form the of

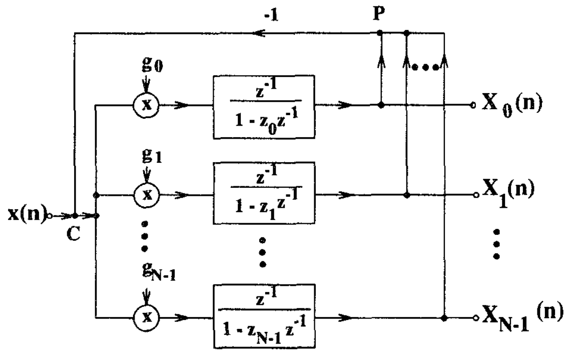

Fig. 1. Feedback-based DFT filter-bank

$$\begin{aligned} T_m(z) &= \frac{H_m(z)}{1+\sum_{n=0}^{N-1} H_n(z)} \\ &= \frac{\frac{g_m z^{-1}}{1-z_m z^{-1}}}{1+\sum_{n=0}^{N-1} \frac{g_n z^{-1}}{1-z_n z^{-1}}}, \quad m=0,1,...,N-1 \end{aligned} \qquad (1)$$

where

$$z_m = e^{j\varphi_m}, \; g_m = \frac{z_m}{\prod_{n=0,n\neq m}^{N-1}\left(1-z_n z_m^{-1}\right)}, \quad m=0,1,...,N-1. \qquad (2)$$

The resonator poles can be located arbitrarily along the unit circle, only the case of multiple poles is to be avoided. For a DFT filter-bank the poles are the Nth roots of unity, therefore its parameters have simpler form

$$z_m = e^{j\frac{2\pi}{N}m}, \quad g_m = \frac{z_m}{N}, \quad m=0,1,...,N-1. \qquad (3)$$

The value of the transfer function (1) is zero if $z = z_n$, $n = 0, 1, ..., N-1$, except the case of $n = m$ when it equals 1. If the input sequence of this filter-bank consists of sinusoids of frequency corresponding to z_n ($n = 0, 1, ..., N-1$), then it will separate the mth component and measure it without any systematic error. However, if the frequency of the mth component differs from the pole frequency of the mth resonator, then a systematic magnitude and phase error will appear. For the DFT case the magnitude error can be expressed by

$$\left|\frac{\sin(N\pi\Delta f T)}{N\sin(\pi\Delta f T)}\right|, \qquad (4)$$

where Δf and T stand for the frequency error and for the sampling time, respectively. Equation (4) is easy to derive since for the DFT case (1) can be expressed in a much simpler form of

$$T_m(z) = \frac{1}{N}\left(1-z^{-N}\right)\frac{z_m z^{-1}}{1-z_m z^{-1}}. \qquad (5)$$

The phase of (5) can also be directly derived. It turns out that the phase is practically linear except the unavoidable phase jumps of π at the resonator pole positions of index n ($n \neq m$). The presence of this linear phase shift may cause serious measurement problems also for small Δf values. This problem is well-known also in the high precision measuring bridges, where due to the frequency-dependent parasitic components the balanced state of the bridge will differ from that of the nominal frequency case [9].

To reduce these errors due to the mismatch of the input and pole frequencies the transfer functions of (1) should be modified. A better performance can be achieved if the magnitude and the phase approximate in a wider frequency range one and zero (or at least a constant value), respectively. A possible solution to this approximation is the classical windowing technique (see e.g. [1]) widely used in spectrum analysis. In the filter-bank of Fig. 1 windowing can be performed at the parallel output as a simple linear combination of the neighboring channels. The

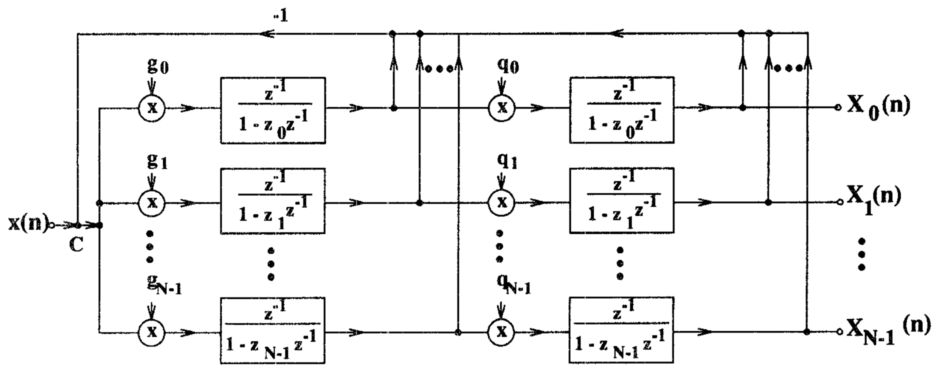

Fig. 2. Feedback-based filter-bank with resonator multiplicity of 2

conventional Hanning window (see e.g. [1]) requires the linear combination of the $(m-1)$th, mth and $(m+1)$th channels. Windowing, however, reduces the frequency resolution, i.e. to provide the same selectivity the order of the system on Fig. 1 should be increased. Unfortunately the increased order will destroy the improvement due to windowing. The next Section will introduce an alternative solution which at the prize of higher system order can improve the magnitude and phase responses of the channels without affecting the frequency resolution.

III. CHARACTERIZATION OF THE RESONATOR-BASED FILTER-BANKS

The single-input multiple-output (SIMO) filter-bank of Fig. 1 can serve as a basic building block in very many practical applications. Using the parameters of (3) the system operates as a DFT filter-bank which time recursively evaluates the DFT of the last N input samples and simultaneously provides the DFT components, as well. The application of the frequency sampling (FS) method is simply the calculation of the linear combination of these DFT components. The weights applied are sampled values of the transfer function to be implemented. The sampling must be performed at the resonator pole frequencies where (1) equals one. The order of the attainable FIR filters is $N-1$. The system of Fig. 1 provides a one-step delay because the channel outputs are the outputs of the delay elements of the resonators, therefore a "leading" z^{-1} is also present in the transfer functions of the filters. This delay can be avoided if the output is calculated as the linear combination of the inputs of the delay elements. The parallel nature of channel outputs enables the simultaneous application of several sets of weights, i.e. the implementation of several filters having the same input.

The first step toward the generalization of the original FS method is the application of arbitrarily located resonators (see (2)). If the input signal consists of components with frequencies corresponding to these poles then similar overall behavior can be expected as in the DFT case. The error caused by the frequency mismatch has similar nature as described above. The only disadvantage is the size of the dynamic range required during the transient phase (first N samples) of the operation if the location of the resonator poles is highly asymmetrical. The second step of the generalization is the introduction of common poles into the transfer functions of (1). If p_n, $n=0,1,...,N-1$, denotes the poles to be implemented then equation (2) must be replaced by

$$z_m = e^{j\varphi_m}\ , \quad g_m = z_m \frac{\prod_{n=0}^{N-1}\left(1-p_n z_m^{-1}\right)}{\prod_{n=0,n\neq m}^{N-1}\left(1-z_n z_m^{-1}\right)}\ , \qquad m=0,1,...,N-1\ . \tag{6}$$

In [4] a method is described to relate the poles of the overall system and those of the resonators. The application of this relation simplifies the final structure and at the same time helps to avoid zero-input limit cycles. As a further step of the generalization the case of multiple resonator poles should be mentioned. The corresponding structure with resonator multiplicity of 2 (in every channel) is given in Fig. 2. The overall system order is $2N$. The structure is in complete correspondence with the Hermite interpolation. Its parameters can be derived from the transfer function of the form of

$$\begin{aligned} T_m(z) &= \frac{H_m(z)}{1+\sum_{n=0}^{N-1} H_n(z)} \\ &= \frac{\frac{g_m z^{-1}}{1-z_m z^{-1}}+\frac{g_m q_m z^{-1}}{(1-z_m z^{-1})^2}}{1+\sum_{n=0}^{N-1}\left(\frac{g_n z^{-1}}{1-z_n z^{-1}}+\frac{g_n q_n z^{-1}}{(1-z_n z^{-1})^2}\right)}\ , \\ & \qquad m=0,1,...,N-1 \end{aligned} \tag{7}$$

where

$$z_m = e^{j\varphi_m}\ , \quad g_m = \frac{1-Q_m}{G_m} z_m\ , \quad q_m = \frac{1}{1-Q_m}\ , \qquad m=0,1,...,N-1\ , \tag{8}$$

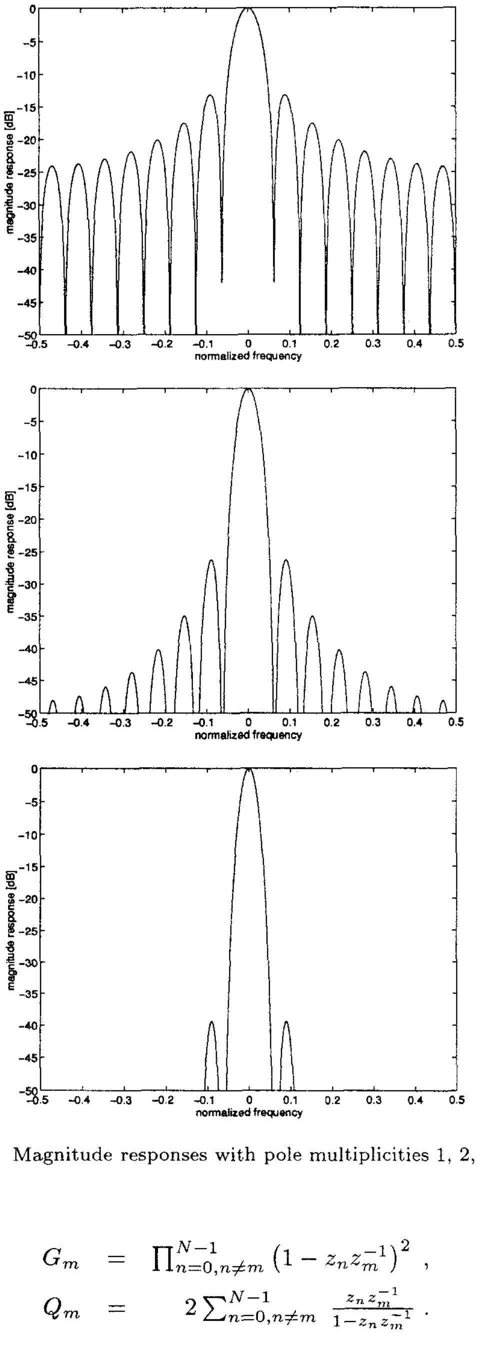

Fig. 3. Magnitude responses with pole multiplicities 1, 2, and 3

with

$$
\begin{aligned}
G_m &= \prod_{n=0,n\neq m}^{N-1} \left(1 - z_n z_m^{-1}\right)^2 , \\
Q_m &= 2\sum_{n=0,n\neq m}^{N-1} \frac{z_n z_m^{-1}}{1-z_n z_m^{-1}} .
\end{aligned} \qquad (9)
$$

This derivation is based on the fact that (7) should equal to one, and its derivative to zero at $z = z_m$. If the resonator poles equal to the Nth roots of unity then (8) will have a much simpler form of

$$
z_m = e^{j\frac{2\pi}{N}m} , \quad g_m = \frac{z_m}{N} , \quad q_m = \frac{1}{N} . \qquad (10)
$$

The development of the higher multiplicity cases is straightforward, the only difference is that the higher order derivatives are also to be considered. For FIR filter implementation this generalization of the FS methods means that not only the transfer function values at $z = z_m$, $m = 0, 1, ..., N-1$ are taken into account but its first, second, etc. derivatives, as well. The filter output is composed as the linear combination of the output of each resonator, i.e. each channel contributes to the filter output with two complex weights. If filter poles are also needed, i.e. the transfer functions of (7) should implement common poles, then the g_m and q_m $(m = 0, 1, ..., N-1)$ values can be calculated similarly as in the case of the Lagrange structure.

The filter-banks with multiple resonator poles offer a wide variety of possible systems since the multiplicity of these poles can be different from channel to channel. The system of Fig. 2 is only a first attempt to improve the performance of sine wave measurements. If the resonator pole frequencies coincide with that of the signal components, then the direct utilization of outputs $X_m(n)$, $m = 0, 1, ..., N-1$ will provide better noise immunity. For the DFT case this fact is illustrated in Fig. 3 where the magnitude responses with resonator pole multiplicities of 1, 2, and 3 are provided. If there is a difference Δf in the frequencies, then the sum of the two resonator outputs of the very same channel will give better performance, since this channel transfer function will equal to 1 at the resonator pole position and simultaneously its derivative is forced to be zero. This latter means that at this point the derivative of the magnitude and the phase is zero and therefore the systematic error in the case of frequency difference is less than with the Lagrange structure. The actual magnitude and phase error can be calculated using standard methods from equation (7).

IV. ADAPTIVE FOURIER ANALYSIS WITH MULTIPLE RESONATORS

The structures investigated above can be efficiently utilized if the center frequencies of the measuring channels are "synchronized" to a certain extent to that of the signal components to be measured. If a reference signal is available to directly control the resonator pole positions then this synchronization can be solved. Typical examples are the measurements of harmonically related sinusoids where the fundamental frequency is time-varying but can be measured with certain accuracy. Since this accuracy is usually very limited due to the dynamics of the frequency changes, the better performance near to the center frequency locations can be of real importance.

If there is no available reference signal then the concept of the Adaptive Fourier Analyzer (AFA) described in [6] can be used and generalized. This analyzer is a very efficient adaptive filter-bank, which tries to lock to the fundamental frequency of the signal and suppress all the harmonically related component except one as it is the case of the time-recursive DFT. The system adapts not only its parameters but also its structure. This latter is performed simply by accommodating as many channels into the common loop as many harmonically related frequency positions can be located to the interval from zero to the sampling frequency. Obviously for lower fundamental frequencies the system order $N(n)$ will increase and for higher ones decrease. The complete adaptation algorithm can be directly utilized also for the case of this Hermite interpolation structure. The only problem to be solved is the selection of the optimal channel characteristics for the fundamental frequency to provide fast convergence. Some early simulations show that the performance of the system in Fig. 2 seems to be very similar to that of the Lagrange structure if both solutions apply the same number of channels. This latter means that the considerable increase of the system order affects the convergence speed only to a small extent. The adaptation procedure is based on the output signal $X_1(n)$. The performance with combined output, i.e. with the combination of the two resonator outputs of the fundamental channel is under investigation.

V. CONCLUSIONS

In this paper the possible generalizations of the frequency sampling (FS) method were investigated. It was shown that if signals consisting of sinusoidal components are to be measured the structure based on the classical Hermite interpolation technique provides better performance. The prize to be paid for this improvement is the increase of the filter-bank order, however, especially for the case of uniform resonator pole distribution this fact can be tolerated. The suggested system can also be used to implement arbitrary FIR and IIR filters similarly to the structures described [4] and [7], but additionally, with the generalization of the frequency sampling method, higher approximation power can be concentrated to certain frequency locations. The Adaptive Fourier Analyzer algorithm [6] can be used to adapt the fundamental frequency of this new system, as well.

REFERENCES

[1] L.R. Rabiner, B. Gold, *Theory and Application of Digital Signal Processing*, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[2] K.E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley & Sons Inc., New York, 1989.

[3] G. Péceli, "A Common Structure for Recursive Discrete Transforms", *IEEE Trans. on Circuits and Systems*, Vol. 33, pp. 1035-1036, Oct. 1986.

[4] G. Péceli, "Resonator-Based Digital Filters", *IEEE Trans. on Circuits and Systems*, Vol. 36, pp. 156-159, Jan. 1989.

[5] M. Padmanabhan, K. Martin, "Resonator-Based Filter-Banks for Frequency-Domain Applications", *IEEE Trans. on Circuits and Systems*, Vol. 38, pp. 1145-1159, Oct. 1991.

[6] F. Nagy, "Measurement of Signal Parameters Using Nonlinear Observer Theory", *IEEE Trans. on Instrumentation and Measurement*, Vol. 41, pp. 152-155, Febr. 1992.

[7] M. Padmanabhan, K. Martin and G. Peceli, *Feedback-Based Orthogonal Filters*, Kluwer Academic Publishers, Boston/London/Dordrecht, 1996.

[8] K. Godfrey, *Perturbation Signals for System Identification*, Prentice-Hall International (UK) Ltd., 1993.

[9] L. Schnell (Ed.), *Technology of Electrical Measurements*, John Wiley & Sons, 1993.