Confluence and Critical Pair Analysis in Graph Transformation Systems

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Foundations of Model Driven System Development

What is Confluence?

G
H1
H2
H11
H12
H21
H22

Do transformations exist which lead to one common graph?

X

GTS is called confluent, if for all (typed) graph transformations \( G \xrightarrow{H_1} \) and \( G \xrightarrow{H_2} \) there is a (typed) graph \( X \) together with (typed) graph transformations \( H_1 \xrightarrow{X} \) and \( H_2 \xrightarrow{X} \).
• **Global determinism:** for each pair of terminating (typed) graph transformations $G \Rightarrow^* H_1$ and $G \Rightarrow^* H_2$ with the same source graph also the target graphs $H_1$ and $H_2$ are equal or isomorphic,

• $G \Rightarrow^* H$ is called **terminating** if no (typed) graph production in the GTS is applicable to $H$ any more.

• Each confluent (typed) graph transformation system is globally deterministic.

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**How to show confluence?**

• A system is **locally confluent** if all conflicting pairs of transformations are confluent.

• A local confluent and terminating system is **confluent**.

• To check local confluence it is enough to check all **critical pairs**, i.e. minimal conflicting situations.
Local Church-Rosser Theorem

- $G \to H_1$ and $G \to H_2$ independent
- Then $H_1 \to X$ and $H_2 \to X$ exist
- shown for labeled graphs
- can be rephrased for typed graphs

Local confluence

- Local confluence, where the given (typed) graph transformations $G \Rightarrow H_1$ and $G \Rightarrow H_2$ are direct (typed) graph transformations, but $H_1 \Rightarrow^* X$ and $H_2 \Rightarrow^* X$ are still general.

- Each terminating and locally confluent (typed) graph transformation system is already confluent.
Critical Pairs

- Two rules $p_1$ and $p_2$ with attributes over $\Sigma \cap T(X)$ and disjoint subsets of $X$
- $K \rightarrow P_1$ and $K \rightarrow P_2$ is a CP candidate if non-independent, $K$ is attributed over $T(X)$
- A critical pair is a minimal CP candidate (minimal graph and maximal)

\[ X \]
\[ P_1 \]
\[ H_1 \]
\[ H_2 \]
\[ G \]
\[ K \]
\[ p_1 \]
\[ p_2 \]

Critical pairs

- A pair $P_1 \leftarrow K \rightarrow P_2$ of direct (typed) graph transformations is called a critical pair, if it is non-parallel independent and minimal in the sense that each item in $K$ has a preimage in $L_1$ or $L_2$ ($K$ can be considered as a suitable gluing of $L_1$ and $L_2$).

- Completeness of critical pairs:
  For each pair of non-parallel independent direct graph transformations $H_1 \leftarrow G \rightarrow H_2$, there is a critical pair $P_1 \leftarrow K \rightarrow P_2$
Critical pairs and local confluence

• Strict confluence: the largest subgraph \(N\) of \(K\), which is preserved by the critical pair \(P_1 \Leftarrow K \Rightarrow P_2\), is also preserved by \(P_1 \Rightarrow^* K'\) and \(P_2 \Rightarrow^* K'\).

• Theorem:
A (typed) graph transformation system GTS is locally confluent, if all its critical pairs are strictly confluent.

Example: Mutual Exclusion
Example: Mutual Exclusion

Example