Pebble shapes and abrasion processes

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The origin of the shapes of stones and other particles formed by water or wind has always attracted the attention of geologists and mathematicians. A classical model of abrasion due to W. J. Firey leads to a geometric partial differential equation representing the continuum limit of the process. This model predicts convergence to spheres from an arbitrary initial form; analogously, the two-dimensional version of the model predicts convergence to circles. The shapes of real stones are, however, not always round. Most notably, coastal pebbles tend to be smooth but somewhat flat, and ventifacts (e.g. pyramidal dreikanters) often have completely different shapes with sharp edges. Inspired by Firey's results, a new PDE is derived in this paper, which not only appears to be a natural mathematical generalization of the PDE, but also represents the continuum limit of a generalised abrasion model based on recurrent loss of material due to collisions of nearby pebbles. Preliminary results suggest that our model is capable to predict a broad range of limit shapes: polygonal shapes with sharp edges develop due to sand blasting (big stone surrounded by infinitesimally small particles), round stones emerge due to collisions with relatively big stones, and flat shapes are the typical outcome in the intermediate case. The results show nice agreement with real data despite the model's simplicity.

Keywords: surface evolution, curvature, ventifact

1. Introduction

One of the best-known quantitative models of the abrasion process of pebbles has been proposed by Firey (1974). He considered pebbles losing small portions of their volumes via successive collisions with a plane representing the underlying ground. If the orientation of the colliding pebbles is assumed to be random with uniform distribution, the continuous limit of this process is surface wear with speed proportional to curvature (in 3D: Gaussian curvature; in 2D: scalar curvature, see Section 2) provided that the initial shape is convex. Not surprisingly, this process quickly abrades sharp vertices with high curvature, and arbitrary initial shapes become spheres in 3D (Andrews, 1999) or circles in 2D (Gage, 1984, Gage & Hamilton, 1986) as they gradually contract to points.

While real pebbles typically do get smooth, they are usually not perfect spheres. Some deviation follows from the discreteness of abrasion (Durian et al, 2006) nevertheless the fact that coastal pebbles tend to be flat (Rayleigh, 1944, Wald, 1990, Lorang & Komar, 1990) calls for a different explanation. Dobkins & Folk (1970) found that pebbles on Tahiti are sometimes assorted by the wave current and flat ones accumulate on the beach. In some other cases, the flatness reflects the stratified microstructure of the material. According to an alternative explanation, initially flat pebbles collide with the waterbed with non-uniformly distributed orientations which probably helps maintaining/increasing their flatness. It was also proposed by Dobkins & Folk (1970) that the presence of sand plays important role in flattening pebbles. Probably all these factors contribute to the formation of observed pebble shapes. The inclusion of some factors is beyond the scope of a simple PDE, nevertheless the effect of 'sand blasting' is easy to approximate in the spirit of Firey's classical model. In Section 3 we generalize Firey's 2D model by considering collisions with nearby pebbles instead of planes. Our general model includes both Firey's model and also abrasion by infinitesimally small sand particles as special cases. The continuous limit of the new model boils down to another PDE: contraction with speed $p\rho+1$ where ρ is curvature, and p is a constant. In particular, sand blasting corresponds to $p=0$, Firey's model to $p\to\infty$. Some predictions of the model and their physical relevance are discussed in Section 4.

2. Firey's model

The 2D version of the abrasion model of Firey (1974) assumes that a convex stone represented by a closed, convex plane curve S repeatedly collides with a straight line S^*. During each collision, a small fracture of the stone gets lost at the point of contact. We consider the process in a coordinate system fixed to S, see reference axis x and reference point O in Fig. 1 (left). Each collision is parameterized by β denoting the angle of the normal vector of S^* relative to x. We assume that β is a uniform random variable on $[0,2\pi]$, i.e collisions with orientation $\beta_0<\beta<\beta_0+\Delta\beta$ happen with mean frequency $f\Delta\beta$ (f is a constant). The average area δ lost by the pebble S is assumed to be independent of the location of the collision.

We consider an infinitesimal piece of length Δs on which the angle ϕ of the inward pointing normal unit vector $\mathbf{n}_\phi$ of S to axis x is $\phi\in(\alpha-\Delta\alpha/2,\alpha+\Delta\alpha/2)$. In a collision, Δs is hit by the plane S^* if $\alpha-\Delta\alpha/2\leq\beta\leq\alpha+\Delta\alpha/2$ (modulo 2π). The mean frequency of such collisions is $f\Delta\alpha$, hence the mean area loss per unit time is $\delta f\Delta\alpha$. The mean speed of wear is then $\delta f\Delta\alpha/\Delta s$ $=\delta f\rho(\alpha)$ where $\rho(\alpha)$ is the curvature of the curve. Thus, in the continuum limit, the abrasion process leads to the geometric PDE

$$\dot{\mathbf{v}} = \delta f\rho(\alpha)\cdot\mathbf{n}_\alpha \qquad (1)$$

where **v** is the position vector of a point on the perimeter with (time-dependent) normal direction α; dot denotes derivative with respect to time t. Hence, abrasion speed is proportional to curvature. The only possible limit shape of the curve (after scaling) is a circle, as already pointed out in Section 1. The same arguments in 3D lead to surface contraction proportional to Gaussian curvature and convergence to spheres.

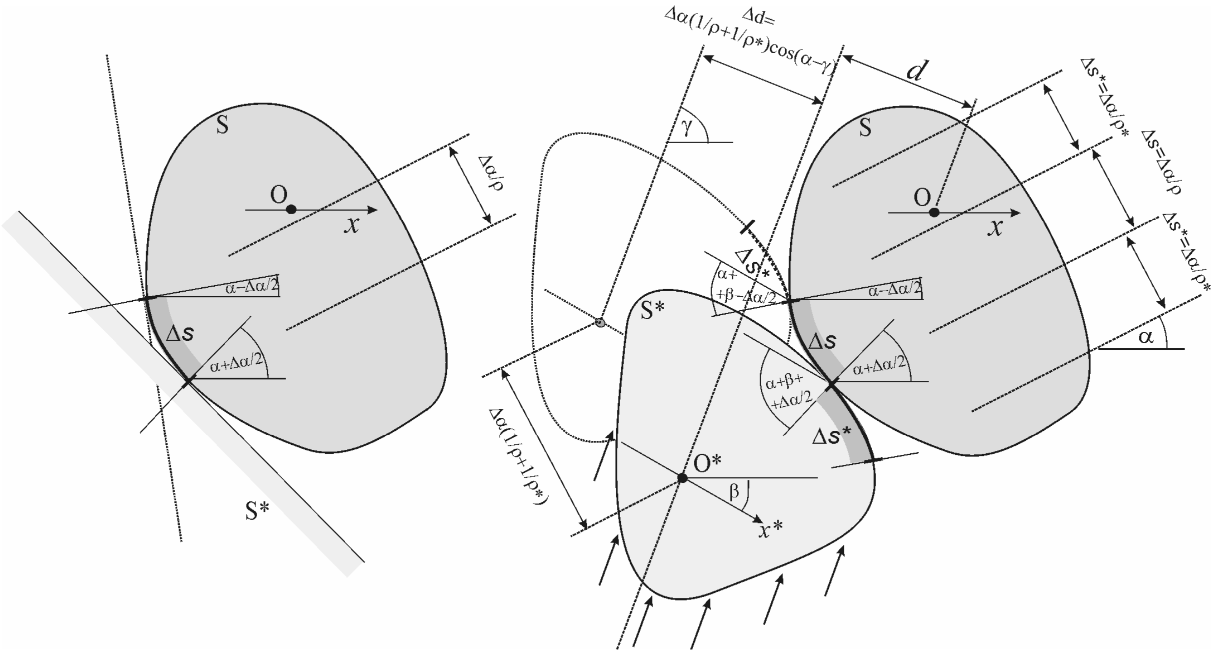

Fig. 1: The abrasion models. Left: Firey's model. Line S^* is shown while colliding to an endpoint of interval Δs (i.e. $\beta=\alpha+\Delta\alpha/2$). S^* colliding to the other endpoint ($\beta=\alpha-\Delta\alpha/2$) is also shown in dashed line. Right: the generalised model. Curve S^* is shown while colliding to both endpoints of interval Δs (one copy in continuous line, the other one in dashed line). Arrows show the direction of motion of S^*.

3. The generalized model

Here we derive a natural, physical generalization of Firey's model, where the straight line S^* is replaced by a convex curve (Fig. 1, right). Let x, x^*, O, O^* denote reference axes and reference points of the two respective curves. We keep the notations of Section 2; in addition, $\rho^*(\phi)$ denotes the curvature of S^*. S^* is assumed to be subject to straight motion relative to S; a collision of the two curves is parameterized by 3 random parameters: β determines the angle between x and x^*; angle γ denotes the direction of motion of S^* relative to x; finally d denotes the distance of the straight orbit of O^* from O. Δs^* is the interval on S^* where the angle of the outward-pointing normal of the curve is $\phi\in(\beta+\alpha-\Delta\alpha/2,\beta+\alpha+\Delta\alpha/2)$ to x^* (Fig. 1, right).

As in Section 2, it is assumed that d, β, γ are uniformly distributed: collisions with γ, β, d lying in given intervals of sizes $\Delta\beta$, $\Delta\gamma$ and Δd, respectively occur with mean frequency

$f\Delta\beta\Delta\gamma\Delta d$. The pebble S gets hit at Δs if the parameters fall in appropriate ranges: S^* needs to be between the two extreme positions shown in Fig. 1 (right) at the instant of the collision. One of these copies of S^* can be transformed to the other by shifting it by distance

$$\Delta s + \Delta s^* = \left(\frac{1}{\rho(\alpha)} + \frac{1}{\rho^*(\alpha - \beta)}\right) d\alpha \tag{2}$$

in direction perpendicular to $\mathbf{n}_\alpha$. For fixed β and γ, this displacement corresponds to an interval in parameter d of size

$$\Delta d(\beta,\gamma) = \begin{cases} \left(\Delta s + \Delta s^*\right)\cos(\alpha-\gamma) = \left(\frac{1}{\rho(\alpha)} + \frac{1}{\rho^*(\alpha-\beta)}\right) d\alpha \cos(\alpha-\gamma) & \text{if } \cos(\alpha-\gamma) > 0 \\ \text{no such interval} & \text{otherwise} \end{cases} \tag{3}$$

The mean frequency of collisions hitting Δs is obtained via the following integral, which is simplified after substitution of (3) (computational details omitted):

$$\begin{aligned} f_{\Delta s} &= f \int_0^{2\pi} \int_{\alpha-\pi/2}^{\alpha+\pi/2} \Delta d(\beta,\gamma) d\gamma d\beta = f \int_0^{2\pi} \int_{\alpha-\pi/2}^{\alpha+\pi/2} \left(\frac{1}{\rho(\alpha)} + \frac{1}{\rho^*(\alpha-\beta)}\right) d\alpha \cos(\alpha-\gamma) d\gamma d\beta = \\ &= 2 f d\alpha \left[\frac{2\pi}{\rho(\alpha)} + \int_0^{2\pi} \frac{1}{\rho^*(\beta)} d\beta\right] \stackrel{def}{=} 2 f d\alpha \left[\frac{2\pi}{\rho(\alpha)} + P^*\right] \end{aligned} \tag{4}$$

Here P^* denotes the length of the perimeter of S^*. The resultant contraction PDE is

$$\dot{\mathbf{v}} = \frac{\delta \cdot f_{\Delta s}}{\Delta s} \cdot \mathbf{n}_\alpha = 2\delta f \left[2\pi + P^* \rho(\alpha)\right] \cdot \mathbf{n}_\alpha \stackrel{def}{=} c\left[1 + \frac{P^*}{2\pi}\rho(\alpha)\right] \cdot \mathbf{n}_\alpha . \tag{5}$$

where c is a constant. Notice that the first (constant) term dominates in (5) if $P^* \ll 1$ (collisions with infinitesimal particles, i.e. sand-blasting), whereas the second one dominates if $P^* \gg 1$. The latter limit provides Firey's original model. In case of real pebbles, P^* should be considered as average size of particles in the examined pebble's environment. For further analysis, (5) is non-dimensionalized:

$$\dot{\mathbf{v}} = c\left[1 + \frac{P^*}{P} \cdot \frac{P\rho(\alpha)}{2\pi}\right] \cdot \mathbf{n}_\alpha \stackrel{def}{=} c\left[1 + \bar{p} \cdot \bar{\rho}(\alpha)\right] \cdot \mathbf{n}_\alpha \tag{6}$$

Here P is the length of the perimeter of S. The scale-free parameter $\bar{p} = P^*/P$ denotes the relative size of nearby stones; $\bar{\rho}(\alpha) = P\rho(\alpha)/(2\pi)$ is scale-free curvature, which depends only on shape but not on size and its 'average' value is 1 for every curve.

We remark that a similar model can be derived in 3 dimensions. The result is a PDE consisting of 3 abrasion terms: a constant term, a term proportional to mean curvature, and another one proportional to Gaussian curvature. One of the 3 terms dominates in the following cases respectively: collisions to infinitesimal particles, lines, and planes.

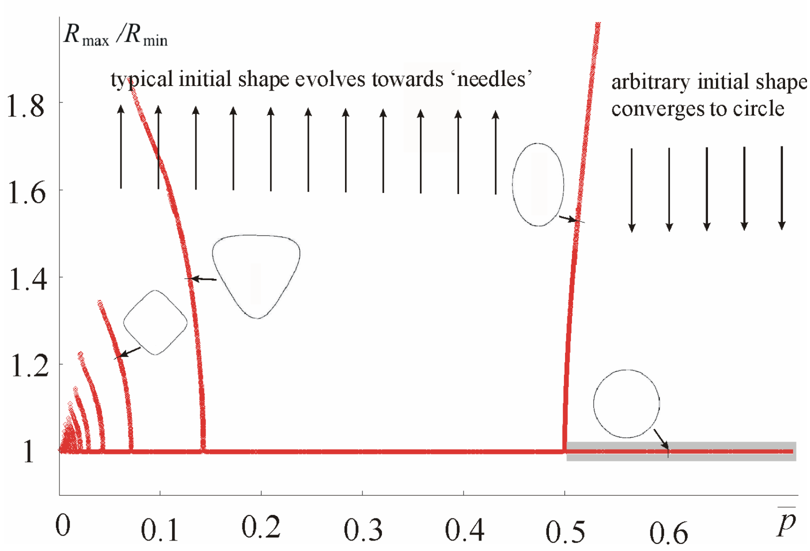

Fig. 2: Invariant shapes of the abrasion model (6) determined numerically by a nonlinear boundary value problem solver (Domokos & Szeberényi, 2004). Horizontal and vertical axes show parameter $\overline{p}$, and the ratio of maximal and minimal distances of the obtained solutions measured from the centroid (R_{max}/R_{min}), respectively. Points with grey background are attracting, the rest are repelling. Circle shapes (R_{max}/R_{min}=1) are attracting if $\overline{p}>1/2$ and repelling otherwise. The branches of solutions with n-fold rotation symmetry are n-1 times unstable; the n=2 branch lies on a separatrix between shapes evolving to circles and 'needles' if $\overline{p}$ is slightly above 1/2.

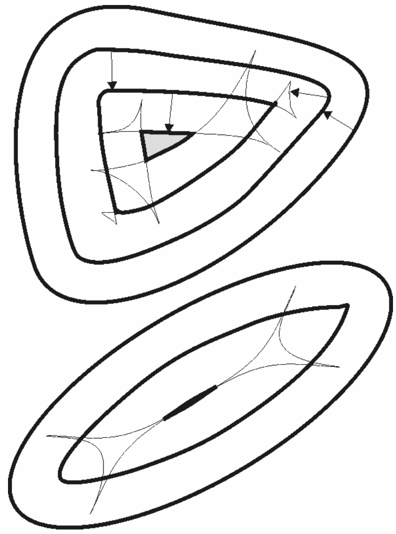

Fig. 3: Typical outcomes of contraction by constant speed ($\overline{p}$=0): triangle (top) and 'needle' shape with flatness growing to infinity (bottom). The latter occurs usually for elongated initial shapes and the former one for 'compact' initial shapes. More precise classification can be given using the maximal inscribed circle of the initial contour (not shown here).

4. Discussion

Based on (6), the following question arises: what is the (typical) outcome of the wearing process at given $\overline{p}$? Analytical results of Andrews (2003) on a similar PDE (namely contraction proportional to $\rho(\alpha)^{p}$) imply that circles are locally attracting if $\overline{p}\in(1/2,\infty)$ and repelling if $\overline{p}\in(0,1/2)$. Numerical simulations suggest that they are globally attracting for big $\overline{p}$ and the length/width ratio of typical initial shape blows up (it becomes 'needle-like'), i.e. there is no well-defined limit shape for small $\overline{p}$. Furthermore, there is an interval of $\overline{p}$ above 1/2, where the final outcome depends on the initial shape (Fig. 2). The two attracting limits predicted by the model agree with real pebble shapes: these are sometimes spheres and flat (although not infinitely flat) shapes in other cases.

The case $\bar{p}=0$ is especially interesting, not only physically (sand blasting) but also mathematically. The PDE becomes hyperbolic: instead of smoothing, typical initial curves develop cusps, beyond which the stones' shapes are 'weak solutions' of the PDE. In our case, 'needles', as well as arbitrary triangles with sharp edges are attracting solutions (Fig. 3); a rich variety of unstable invariant shapes also exists (Pegden, in preparation). Similar shapes (so-called ventifacts) can be created by wind abrasion. Ventifacts shaped by wind-driven sand most often look like polyhedra; the most characteristic ones are tetrahedral dreikanters. Thus, our model offers a new and fairly simple explanation for dreikanter generation, while it also embraces the classical pebble wear process.

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