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Floating body problems in two dimensions[1]

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Abstract

Stanislav Ulam asked if the sphere is the only object floating in neutral equilibrium in every orientation and negative answer was provided recently. Here, several related problems are discussed. The same question is asked for two-dimensional objects whose centroid is pinned and it is demonstrated that the answer is similar to the case of freely floating bodies. We also discuss the minimal number of equilibria of homogenous planar floating objects (either freely or with pinned centroid) representing duals of Ulam's Floating Body Problem. The non-existence of shapes with less than four equilibria is proven in special cases including infinitesimal perturbations of a circle however the general question remains open. The paper is complemented with remarks on analogous problems in three dimensions; connections to the family of Four-Vertex theorems are also pointed out.

Key words: equilibrium, floating, monostatic, Floating Body Problem

1. Introduction

The equilibria of floating objects are among the most classical issues of physics, which provide many examples of non-intuitive behaviour. The basic principles of floating were first worked out in the famous book 'On Floating Bodies' of Archimedes [1]. Based on his geometric results on parabolas, Archimedes was also able to determine the resting points of floating paraboloid segments of revolution. He, and recent authors completing his work [2,3] found that such objects have 3 to 7 practically different equilibria depending on shape and density. Even simpler shapes, such as rods, cylinders or the arcimedean solids show unexpectedly complex floating behaviour [4-6]. For example, a homogenous cube may float

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stably with a face or a vertex pointing down depending on its density and it also has an attractive set of non-isolated skew equilibria in certain ranges of the density. The surprisingly complex bifurcation diagrams of simple floating shapes explain the often observed sudden turnovers of slowly melting icebergs [2,7,8].

One of the most beautiful examples of their strange behaviour is pointed out by Stanislav Ulam in Problem 19 of the Scottish Book[3] [9]:

Floating Body Problem (FBP): are the spheres the only convex, homogenous bodies floating in equilibrium in arbitrary orientation?

Despite the simplicity of the question, the FBP remains open. It is known that the sphere is the only one among centrally symmetric bodies of density ρ=1/2 relative to water [10,4], however a recent work [11] suggests that there are probably other neutral shapes in general. Even the two-dimensional analogue of the question

Planar Floating Body Problem: if a long, convex, homogenous prism floats in equilibrium in arbitrary orientation (with horizontal axis), is its cross-section necessarily a circle?

is far from trivial. Counterexamples with density 1/2 haven been known for a long time [12]. There are other special values of ρ at which appropriate *infinitesimal perturbations* of the round cross-section preserve the neutral behaviour of the prism in leading order [13,14], and F. Wegner [15,16] could also prove recently that these correspond to real counterexamples. The focus of this paper is on related questions in two dimensions. For simplicity, the above two-dimensional problem is referred to as FBP, and the term 'cross-section of a prism' is replaced by 'planar object'. We remark that different problems called Floating Body Problem also exist in the literature.

In Section 2, basic principles of 'planar' floating as well as some of the above cited results are recalled. Section 3 is devoted to a modified version of the FBP, which has received no attention before:

[3] The Scottish Book is available at http://banach.univ.gda.pl/pdf/ks-szkocka/ks-szkocka3ang.pdf

Problem 1: are there convex planar shapes other than the circle, which are in equilibrium in any orientation if partially immersed in water with the centre of gravity G pinned (allowing only rotations around G)?

Pinned objects also show complex behaviour [5], in which the height *d* of G (above the water surface) plays the role of parameter ρ of free floating. Here we show that for infinitesimally perturbed circles, the behaviour of pinned objects is identical to freely floating ones (*Theorem 1*), implying the existence of infinitesimal deformations of a pinned circle preserving its neutrality. Weather or not these correspond to finite neutral deformations is beyond the scope of the current paper, however the similarity to free floating and analogous results in the latter case [15,16] suggest that they probably do. As a further similarity, we show in *Theorem 2* that there is a rich variety of neutral shapes in the case *d*=0 (being analogous to ρ=1/2).

The FBP and Problem 1 refer to shapes with 'maximal' number of equilibria however the minimal number is also an interesting question. Obviously, every object has at least one stable and one unstable balance point corresponding to the global extrema of its potential energy. Nevertheless, one may ask the

(Planar) Monostatic Floating Body Problem (MFBP): are there convex, homogenous planar shapes of relative density ρ, which have only one stable equilibrium?

as well as its analogue to pinned objects:

Problem 2: are there convex, homogenous planar shapes with only one stable equilibrium, if their centre of gravity G is pinned at distance d above water surface?

Monostatic objects (for an exact definition, see Section 3) return to their unique stable restpoint regardless of the initial position; the ability of spontaneous self-righting is important for many technical tools such as ships or spacecrafts, and also for some animals with rigid shells [17]. Previously, monostatic objects sitting on solid, horizontal surfaces have been widely investigated. One can easily construct such shapes with inhomogenous mass-distribution similar to 'weebles'; even a (non-regular) tetrahedron may be monostatic as proved but not published by John Conway, see [18]. Objects with homogenous mass-distribution may also be monostatic: the simplest such *polyhedron* is due to Conway & Guy [19]. There are even 'mono-monostatic' 3D shapes with only 1 stable and 1 unstable equilibrium [20,21]. In contrast, no convex, homogenous object has this property in two-dimensions [22]; moreover, the latter is equivalent [20] of the Four-vertex theorem, stating that each plane curve has at least four vertices, i.e. local extrema of the curvature [23]. Resting on a solid surface corresponds to floating in the limit $\rho \to 0$ or $\rho \to 1$. Hence negative answer to the planar MFBP would also correspond to a novel physically inspired generalisation of the Four-vertex theorem, which is a classical result of differential geometry with countless existing, purely geometrical extensions.

In Section 4, the nonexistence theorem of [22] is summarized with a modest generalisation to objects with inhomogenous density $\rho(r)$ depending only on distance r from the object's centre of gravity (*Theorem 3*). It is also shown that certain $\rho(r)$ functions allow monostatic floating (*Theorem 5*) whereas others exclude it (*Theorem 4*). Nevertheless the case of our primary interest (ρ=*constant*) is not among these simple ones.

The most straightforward connection between solutions of the neutral floating problems (*FBP*, *Problem 1*) and the monostatic ones (*MFBP*, *Problem 2*) is that infinitesimal perturbations of the former ones are potential examples of the latter. Such perturbations are the subject of Section 5. After solving the special cases $\rho \approx 0$, $\rho \approx 1/2$ and $d \approx 0$ (*Theorems 6-8*), further partial negative answer to the planar MFBP and *Problem 3* for $\rho \neq 1/2$ and $d \neq 0$, respectively, are given by showing that infinitesimal perturbations of a circle are never monostatic (*Theorems 9-10*).

The paper is closed by a short discussion. Despite the negative results of Section 5, we outline some promising ways to construct monostatic floating bodies. Preliminary results concerning spatial monostatic floating body problems, as well as some other related mechanical questions are also mentioned.

2. Basic principles of floating

In this Section, we review simple results on ‘planar’ floating, i.e. on floating of long cylinders with horizontal axis. We also summarize some results concerning the FBP following [12], and [4].

According to Archimedes’ law, a planar object Ω of area A, and relative density ρ floats such that a portion of area $A_B=\rho A$ is immersed in the water. Furthermore to float in equilibrium, the line joining the centre of gravity G of the object and that of the submerged part (centre of buoyancy, B) has to be vertical. The sizes of the gravitational and the buoyancy force are $\rho A g$; assuming that the gravitational constant is $g=1$, the potential energy of Ω is

$$U_\rho(\alpha)=\rho A\cdot h_\rho(\alpha)=A_B h_\rho(\alpha)\overset{def}{=}S(\alpha) \tag{1}$$

where angle α determines the direction of the water line relative to a reference axis x of the floating object (Figure 1), $h_\rho(\alpha)$ denotes the height difference between B and G, and thus S is the vertical component of the geometric ‘first order moment’ of the submerged part to G. Equilibria correspond to extrema of U, i.e. those of $h_\rho(\alpha)$. *Neutral floating* for arbitrary α occurs if $h_\rho=constant$, hence if B moves on a circle centred at G as α is varied.

Let B^* denote the centroid of the portion above the water-line. Then, since this part is of area $(1-\rho)A$, B^* becomes the centroid of the submerged part at orientation $\alpha+\pi$ for density $1-\rho$; according to the definition of centroids, the height difference between G and B^* (i.e. $h_{1-\rho}(\alpha+\pi)$) satisfies

$$h_{1-\rho}(\alpha+\pi)=h_\rho(\alpha)\rho/(1-\rho). \tag{2}$$

From (1) and (2),

$$U_{1-\rho}(\alpha+\pi)=U_\rho(\alpha). \tag{3}$$

Thus, there is duality between the equilibria for densities ρ and 1-ρ, hence it is enough to discuss the $\rho\le 1/2$ case as also noted by many authors previously. In particular, if $\rho=1/2$, the U function is π-periodic and an equilibrium at α implies the existence of another one of the same stability type at $\alpha+\pi$.

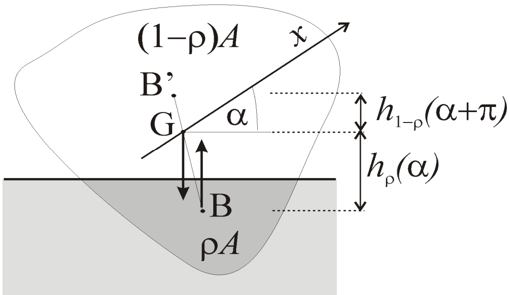

Figure 1: planar floating object in coordinates fixed to the liquid. *x* is a reference axis of the object; B, B*, and G are centroids of the immersed part, the part above the liquid surface, and the total object, respectively. The object in the figure satisfies Archimedes' law but it is not in equilibrium (BG is not vertical).

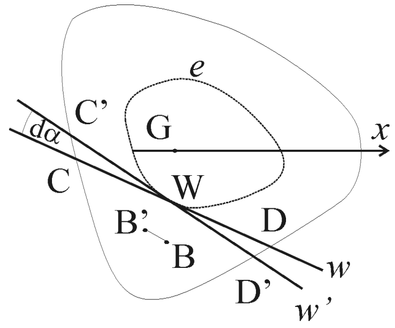

Figure 2: floating object in coordinate system fixed to itself. The two water lines *w*, *w*' meet at angle dα. For further notations, see text.

Consider now a pair of water lines *w* and *w*' satisfying Archimedes' law, and meeting at an infinitesimal angle dα (Figure 2). We assume that each water line crosses the perimeter of the object at 2 points, which is true for convex shapes and also for 'not too concave' ones. Similarly to previous works, we refer to this property as *sufficient convexity*. Since the portions of the objects cut by the two water lines are equal, the triangles CC'W and DD'W in Figure 2 are also of equal area; thus the crossing point W of the lines is within O(dα) of the midpoint of the waterlines; hence, as α is varied, the water line rotates around its current midpoint (combined with some shift parallel to itself). The set of midpoints for $0 \le \alpha < 2\pi$ determines a curve *e* (more precisely a wavefront of total rotation 2π, which may contain cusp singularities) referred to as *water envelope* of the object (or 'floating body of the object' [24]); the water lines are tangents of the water envelope. For ρ=1/2, the waterlines at α and at α+π coincide, thus the water envelope reduces to a wavefront of total rotation π, which is twice circuited by W as α goes from 0 to 2π.

Now let us assume that an object of density ρ floats neutrally! Then, B lies on a circle of some radius *r* around G for every α. For an infinitesimal perturbation dα of α, the displacement of B has to be constant *r*dα; it can also be expressed in terms of the half-length *l* of the waterline CD: the areas of triangles CC'W and DD'W are $l^2 d\alpha/2+O(d\alpha^2)$ and the distance between their centroids is $4/3l+O(d\alpha)$. Thus the displacement of the centre of buoyancy B is $rd\alpha=2l^3 d\alpha/(3\rho A)$ yielding

$$l=(3\rho Ar/2)^{1/3}=constant. \qquad (4)$$

This condition is necessary, and for ρ=1/2 also sufficient: if l satisfies (4), then B moves on a circle of radius r; the π-periodicity of the problem in α implies that the circle is invariant to rotation by π around G; thus it is centred at G. Hence, for ρ=1/2 one can take an arbitrary wavefront of total rotation π as water envelope, and choose an almost arbitrary l (the convexity requirement gives some lower bound on l). On each tangent w of the water envelope one considers the points C, and D of distance l from the point of tangency W. By walking around the water envelope once, the two points cover two halves of the perimeter of an object neutrally floating at ρ=1/2 (see e.g. Figure 3). For ρ≠1/2, the water envelope is of total rotation 2π, and both points cover the whole perimeter while the tangent goes around the water envelope once; however for an arbitrarily chosen initial water envelope, C, and D typically determine different contours, i.e. the above procedure does not work. Beyond that, notice that (4) is not sufficient since the circle on which B travels need not be centred at G. Thus, even if for some l and an initial water envelope the two points determine the same contour, the final object need not be neutral. For some consequences, see the last section. We remark that the only known non-trivial solutions of the FBP for ρ≠1/2 [15,16] have rotational symmetry, which ensures the coincidence of G with the centre of the circle on which B lies.

3. Floating with pinned centroid

Here we consider a homogenous, sufficiently convex (see definition in Section 2) planar object *fixed at its centre of gravity* G at height d above the water surface, but freely rotating around G. The boundary is given by the function $R(\phi)$ in a polar coordinate system centred at G. We assume that for arbitrary orientation, the object is partially submerged in the water (i.e. $|d|$ is not bigger than the (minimal) distance $\min R(\phi)$ of the perimeter from G.). If the object is completely under or above the liquid surface, its behaviour is trivially neutral.

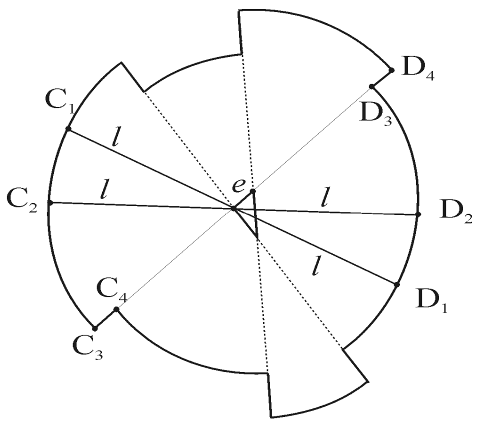

Figure 3: simple example of an object, which floats neutrally in arbitrary orientation if ρ=1/2. The triangle *e* can be considered as a wavefront of total rotation π; we choose it as water envelope; *l* is arbitrary. The emerging boundary consists of line and circle segments, the latter are centred at the triangle's vertices. The boundary is not convex, yet it is just at the limit of being 'sufficiently convex' as defined in the text. C_iD_i are four water lines of length *l*, each one touching *e* at its midpoint.

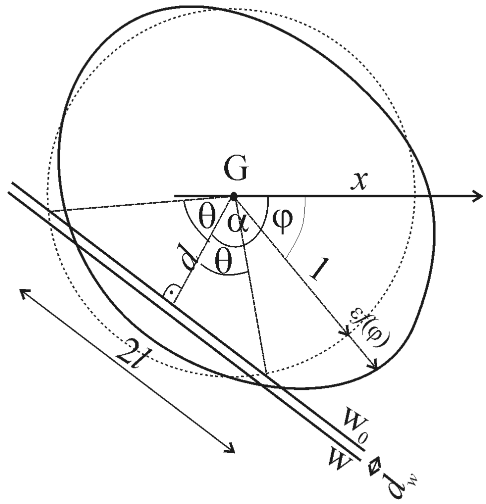

Figure 4: infinitesimally perturbed circle. The line w_0 is at distance *d* from G, and it cuts off a piece of area ρA from the *unperturbed circle*, whereas *w* cuts of a piece of area ρA from the *perturbed object*.

This situation shows much similarity to free floating: equilibria occur if and only if the centroid B of the submerged part is exactly under G. The potential energy is somewhat different of (1). The forces acting at G have zero displacement as α is varied, thus only the mechanical work of the buoyancy force is responsible for variations of the energy. The emerging formula of the potential energy is

$$U_d(\alpha) = A_B(\alpha) h_d(\alpha) \overset{def}{=} S_w(\alpha) \tag{5}$$

where $A_B(\alpha)$ is the area of the submerged part and h_d is the distance of B *from the water line* w_0. Thus S_w is the geometric first moment of the submerged part to w_0. Similar to the duality in parameter ρ of free floating (Section 2), here the existence of an equilibrium with $d=d_0$ at α imply the existence of another one with $d=-d_0$ at $\alpha+\pi$. Thus, it is enough to consider the $d\geq 0$ case.

One apparent similarity between floating and 'pinned floating' is that the equilibrium condition is the same in both cases: BG must be vertical. Hence a floating equilibrium with given ρ always corresponds to a 'pinned' equilibrium with some d. Even more can be stated in case of nearly round objects.

Theorem 1: for arbitrary $d<1$, there is ρ such that the potential $U_d(\alpha)$ of an infinitesimally perturbed unit circle of density ρ whose centroid is pinned at height d above the water surface equals the potential $U_\rho(\alpha)$ of the same shape floating freely in leading order (modulo a constant term).

Proof: we consider a shape determined by

$$R(\phi)=1+\varepsilon f(\phi) \quad (\varepsilon<<1) \tag{6}$$

(Figure 4). If the object is pinned, the water line w_0 is at distance d from G, and the submerged part of the object's perimeter corresponds to the interval $(\alpha-\mathrm{acos}(d)+O(\varepsilon),\alpha+\mathrm{acos}(d)+O(\varepsilon))$ of the polar variable ϕ. We introduce $\theta=\mathrm{acos}(d)$ and express the potential based on (5) as

$$U_d(\alpha)=S_{d0}+\int_{\alpha-\theta}^{\alpha+\theta}\varepsilon f(\phi)\left(\cos(\phi-\alpha)-\cos\theta\right)d\phi+O(\varepsilon^2) \tag{7}$$

where S_{d0} is the first moment of the submerged part for unperturbed circle ($f(\phi)\equiv 0$).

We now develop a similar formula for free floating. ρ is chosen such that the water line of a floating, unperturbed unit circle is w_0. After perturbation, the area A_B cut off by w_0 is

$$A_B=\rho A+\int_{\alpha-\theta}^{\alpha+\theta}\varepsilon f(\phi)d\phi+O(\varepsilon^2) \tag{8}$$

Since the immersed part of a freely floating object is constant ρA and the half-length of w_0 is

$$l=\sin(\theta)+O(\varepsilon), \tag{9}$$

the water line w of the floating perturbed circle is at distance

$$d_w = \frac{A_B}{2l} = \frac{\varepsilon}{2\sin(\theta)} \int_{\alpha-\theta}^{\alpha+\theta} f(\phi)d\phi + O(\varepsilon^2) \tag{10}$$

from w_0. The potential energy of the object is

$$U_\rho(\alpha) = S_{\rho 0} + \int_{\alpha-\theta}^{\alpha+\theta} \varepsilon f(\phi)\cos(\phi-\alpha)d\phi + O(\varepsilon^2) - (l_w + O(\varepsilon))d_w(d + O(\varepsilon)) \tag{11}$$

(cf. (1)). $S_{\rho 0}$ denotes the potential energy of a circle. The second term represents the effect of the perturbation if the water line is fixed at w_0, and the third one reflects the fact that a segment of size $(l_w+O(\varepsilon))d_w$ ceases to be immersed due to the displacement of the water line. Equations (8),(11) and the definition $\theta=\text{acos}(d)$ yield

$$\begin{aligned} U_\rho(\alpha) &= S_{\rho 0} + \int_{\alpha-\theta}^{\alpha+\theta} \varepsilon f(\phi)\cos(\phi-\alpha)d\phi - 2\sin\theta \frac{\varepsilon}{2\sin\theta}\int_{\alpha-\theta}^{\alpha+\theta} f(\phi)d\phi\cos\theta + O(\varepsilon^2) = \\ &= S_{\rho 0} + \varepsilon \int_{\alpha-\theta}^{\alpha+\theta} f(\phi)(\cos(\phi-\alpha) - \cos\theta)d\phi + O(\varepsilon^2) \end{aligned} \tag{12}$$

The energies (12) and (7) are equal modulo constant and $O(\varepsilon^2)$ terms. □

Due to *Theorem 1*, we immediately have

Corollary 1: there is an everywhere dense set of values d in the interval 0≤d<1 for which an appropriate infinitesimal perturbation of a unit circle preserves its neutrality although for randomly chosen d, this is typically not the case.

Proof: The corollary follows from an analogous statement for freely floating infinitesimally perturbed circles (first proved in [13], see also Section 5 of [14]), as well as from *Theorem 1*. We remark that the neutral perturbations are pure k^{th} harmonics where $k\geq 4$, and they exists if

$$\tan(k \cdot \text{acos}d) = k\tan(\text{acos}d). \tag{13}$$

□

In the FBP, a wide family of neutral shapes exist for ρ=1/2, due to the 'self-duality' shown in Section 2. The analogous property of d=0 enables us to construct a wide variety of neutrally behaving pinned shapes:

Theorem 2: a planar objects pinned at G and submerged in water such that d=0, floats neutrally in every orientation iff the polar equation of its boundary $R(\phi)$ in coordinate system centred at G is such that $R^3(\phi)$ has only 0^{th} and $2k+1^{th}$ ($k\geq 1$, $k\in\mathbb{N}$) order Fourier components

Proof: if d=0, the submerged part is simply the union of all points, whose polar angle is in the interval $(\alpha-\pi/2,\alpha+\pi/2)$. Thus, the potential energy can expressed as

$$U_0(\alpha)=\frac{1}{3}\int_{\alpha-\pi/2}^{\alpha+\pi/2}R^3(\phi)\cos(\alpha-\phi)d\phi=\frac{1}{3}\int_0^{2\pi}R^3(\phi)F(\alpha-\phi)d\phi \tag{14}$$

where

$$F(x)\overset{def}{=}\max(0,\cos(x)) \tag{15}$$

U_0 is a convolution of the 2π-periodic functions R^3 and F. According to the Convolution theorem, the coefficients of the Fourier series of U_0 in ϕ are simply products of the corresponding coefficients of R^3 and F. Neutral behaviour means that U_0 is constant; i.e. that all its non-constant Fourier components are 0 apart from the arbitrary zero-order term. F has nonzero 1^{st}, and $2k^{th}$ terms but vanishing $2k+1^{th}$ terms if $k\geq 1$; thus R^3 may consist of arbitrary 0^{th} and $2k+1^{th}$ term, and the rest have to vanish.□

We remark that the vanishing 1^{st} term in R^3 guarantees that the centoid coincides with the origo of the coordinate system. Also notice that arbitrary (positive) function R gives rise to a sufficiently convex object if d=0. A nontrivial example is shown in Figure 5.

4. Monostatic rolling and floating objects

This section focuses on *monostatic* objects in two dimensions; this means that their potential energy versus orientation function $U(\alpha)$ is itself monostatic according to:

Definition 4.1: A 2π-periodic, continuous, scalar function $\Psi(x)$ is monostatic (weakly monostatic), if there exit x_1, x_2 such that f is increasing (nondecreasing) in the interval (x_1,x_2) and decreasing (nonincreasing) in $(x_2,x_1+2\pi)$.

We use a simple consequence of this property, to show the non-existence of monostatic objects in certain cases:

Observation 1: if Ψ (x) is monostatic (weakly monostatic), then there exists x_0 such that

$$\Psi(x) \underset{(\geq)}{>} \Psi(x_0) = \Psi(x_0 + \pi) \underset{(\geq)}{>} \Psi(x + \pi) \quad \text{for} \quad x_0 \leq x \leq x_0 + \pi \tag{16}$$

Notice that the weakly monostatic property includes neutral behaviour, indicating the close relation between the two types of questions of this paper.

We now recall a former result on monostatic objects, resting on a solid surface [22]. This situation corresponds to a special limit of floating, since very light objects (relative to water) lie on top of the water surface with negligible immersion, and thus they behave as if placed on a solid surface. The statement is given here in slightly generalized form:

Theorem 3: let r denote distance from an object's centre of gravity. Then, a convex object of strictly positive density $\rho(r)$ (depending only on r) on a solid, horizontal surface is not monostatic.

The steps of the proof follow those for ρ=*constant* in [22]. The reason for summarizing this result is that similar arguments are used for floating bodies in Section 5. The somewhat artificial type of *r*-dependent inhomogenity is discussed to demonstrate the nontrivial connection between rolling and floating, see *Theorems 4, 5* below.

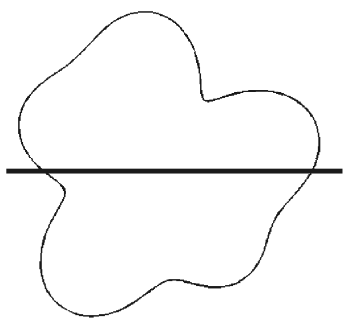

Figure 5: an object, which behaves neutrally if pinned with *d*=0. $R(\phi)=[1+0.4\cos(3\phi)+0.4\sin(5\phi)]^{1/3}$.

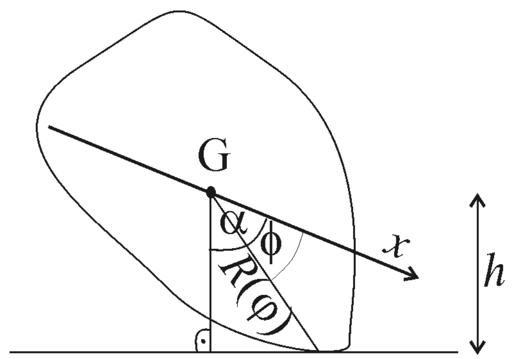

Figure 6: object resting on a solid surface.

Proof by contradiction: the potential energy of a body on a solid surface is $U^*=mgh$ where mg is the weight of the body and h is the height of its centre of gravity G above the surface. For objects given by $R(\phi)$ in polar coordinate system centred at G,

$$U^*(\alpha) = mg \cdot \frac{R(\phi)}{\sqrt{1+\left(\frac{R'(\phi)}{R(\phi)}\right)^2}} . \tag{17}$$

(Figure 6). Here, ϕ denotes the polar angle of the point in contact with the underlying ground, at orientation α (ϕ is usually close to α but they are not equal away from equilibria). The critical points of $U^*(\alpha)$ (i.e. balance points of the object) coincide with those of $R(\phi)$ [22]. Thus, if the object is monostatic, then R has the same property.

The coincidence of the centre of gravity with the origin yields the constraint

$$\int_0^{2\pi}\int_0^{R(\phi)} \rho(r) r \sin\phi dr d\phi = 0 . \tag{18}$$

(as well as the same equation with $\cos\phi$ instead of $\sin\phi$). Since $\sin\phi=-\sin(\phi+\pi)$, (18) is equivalent of

$$\int_0^{\pi} \sin\phi \int_{R(\phi+\pi)}^{R(\phi)} \rho(r) r dr d\phi = 0 . \tag{19}$$

The quantities $\sin\phi$, r and $\rho(r)$ are positive, and $R(\phi)>R(\phi+\pi)$ according to (16) in *Observation 1* for the monostatic function $R()$ with $x_0=0$ (The latter is assumed without loss of generality). Hence the left hand side should be positive leading to contradiction. □

The crucial argument in the proof is the following: the monostatic property implies the existence of a 'thick' and a 'thin' part on the two sides of the centre of gravity. At the same time, the special density-distribution constraints the position of the centre of gravity and these two facts contradict one another. Of course, it is easy to create monostatic objects in case of arbitrary inhomogenous mass distribution. A circle is for example always monostatic if G is not in the geometric centre.

Similar arguments apply in case of the floating problems, and all the nonexistence theorems of Section 5 are based on finding contradiction between these two properties. The potential energy of a floating object is however not as simple as (17). In fact, as we show, there are certain functions $\rho(r)$, for which *Theorem 3* holds to floating objects as well (*Theorem 4*) and others for which it does not (*Theorem 5*). The first statement is demonstrated only for pinned floating, while the second is proved for both the free and the pinned cases.

Let

$$\rho^*(r) \overset{def}{=} \operatorname{acos}\left(\frac{d}{r}\right) - \frac{d}{r}\sqrt{1-\frac{d^2}{r^2}} \tag{20}$$

Then,

Theorem 4: a sufficiently convex object Ω of density

$$\rho(r) = \begin{cases} arbitrary & if \quad r \le \min R(\phi) \\ \rho^*(r) & if \quad r > \min R(\phi) \end{cases} \tag{21}$$

is not monostatic if partially submerged in water with pinned centroid.

Proof by contradiction:

As mentioned above, the statement is trivially true if $|d|>\min R(x)$. This is why the partial submersion ($|d|\le\min R(\phi)$) is assumed. Based on (5), the potential energy for pinned floating at orientation α can be expressed as an integral over the immersed part of Ω:

$$U_d(\alpha) = \int_{\Omega | r\cos(\alpha-\phi)\ge d} [r\cos(\alpha-\phi) - d]\, dA \tag{22}$$

where r and ϕ are the distance and orientation of points of Ω from G. The monostatic behaviour and *Observation 1* imply the existence of α_0 such that $U_d(\alpha)>U_d(\alpha_0)=U_d(\pi+\alpha_0)>U_d(\alpha+\pi)$ for every $\alpha_0<\alpha<\pi+\alpha_0$. Thus, with $\alpha_0=0$,

$$\int_0^{2\pi} U_d(\alpha)\sin\alpha d\alpha = \int_0^{\pi} \left(U_d(\alpha) - U_d(\alpha+\pi)\right)\sin\alpha d\alpha > 0\,. \tag{23}$$

Substitution of (22) into the left side of (23) and changing the order of integration leads to

$$\int_0^{2\pi} \int_{\Omega | r\cos(\alpha-\phi)\ge d} [r\cos(\alpha-\phi) - d]\sin\alpha dA d\alpha = \int_{\Omega} \int_{\phi\text{-acos}\left(\frac{d}{r}\right)}^{\phi+\text{acos}\left(\frac{d}{r}\right)} [r\cos(\alpha-\phi) - d]\sin\alpha d\alpha dA > 0 \tag{24}$$

Notice that the constraints $r\cos(\alpha-\phi)\le d$ and $\alpha\in(\phi\text{-acos}(d/r),\phi+\text{acos}(d/r))$ in (24) are equivalent (both restrict the integration to immersed points of Ω). The inner integral in (24) can be expressed in closed form, thereby we obtain

$$\int_{\Omega | r\ge d} \left(r\cdot\operatorname{acos}\left(\frac{d}{r}\right) - d\sqrt{1-\frac{d^2}{r^2}} \right)\sin(\phi) dA > 0\,. \tag{25}$$

Due to (20), (25) simplifies to

$$\int_{\Omega|r\geq d} \rho^*(r) r \sin(\phi) dA > 0 . \qquad (26)$$

Notice that the above integral is zero over the annulus-shaped domain $d \leq r \leq \min R(\phi)$ of Ω simply by its symmetry. We drop this part from (26) and we also add to it another integral over the disc $0 \leq r \leq \min R(\phi)$, which is also 0 by symmetry:

$$\int_{\Omega|r\geq \min R(\phi)} \rho^*(r) r \sin(\phi) dA + \int_{\Omega|r\leq \min R(\phi)} \rho(r) r \sin(\phi) dA > 0 . \qquad (27)$$

According to (21), this means

$$\int_{\Omega} \rho(r) r \sin(\phi) dA > 0 . \qquad (28)$$

The left-hand side of (28) should be zero by the assumption that r is measured from the centre of gravity G. Hence, we found contradiction, and Ω cannot be monostatic. □

Theorem 5: there exist objects of strictly positive density ρ(r), which are monostatic if floating either freely or with pinned centroid.

Proof: we outline the sketch of a simple construction without exact details. Consider a unit circle with two small dents of area εA_1 ($\varepsilon \ll 1$) and a small hill of area εA_2 arranged symmetrically (Figure 7, left), such that convexity is maintained by the perturbations. Let furthermore A_2 be considerably smaller than A_1. Then, if ρ is constant, the centre of gravity of the object is under the centre O of the circle. However one can choose ρ(r) such that $\rho(r)=1$ if $r \leq 1$ but it quickly increases in the range $r>1$. This way, the centre of gravity can be moved to O. Since the object in question is an infinitesimally perturbed circle, its potential energy function U is the same for free and fixed floating by *Theorem 1*. U can easily be constructed with help of equation (12). The unperturbed circle floats neutrally (U=constant), and the 3 local perturbations correspond to 3 additive perturbations of U in leading order (Figure 7, right). If A_2/A_1 is sufficiently small, U is weakly monostatic, i.e. it contains a constant piece and a 'valley'. A further minor modification of $\rho(r)$ can move the centre of gravity slightly above O, which makes the potential energy function monostatic. □

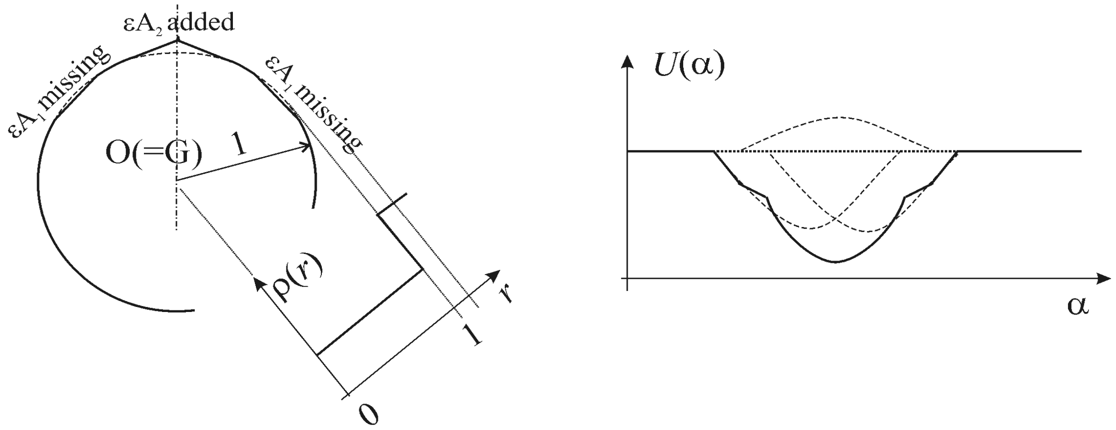

Figure 7, left: a monostatic perturbed circle with inhomogenous mass-distribution ρ(*r*) depending on distance *r* from the centroid. The infinitesimally small perturbations are enlarged. Right: the potential energy of the object. The circle has constant energy (dotted line), the three perturbations correspond to three perturbations of *U* (dashed lines) provided that G remains coincident with O. The energy function (continuous line) is obtained by superposition. *U* is weakly monostatic, the final step of making *U* monostatic is not shown.

5. Nonexistence theorems of the monostatic floating body problem

In Section 4, it was shown that bodies sitting on horizontal surfaces (analogous to floating if $\rho\to 0$) are never monostatic, however the generalisation of this statement to floating objects is far from trivial. Now we prove the non-existence of such shapes in special cases. Analogous results for floating with pinned centroid are also shown. In part 5.1, special values of the parameters ρ and *d* are considered, whereas part 5.2 deals with infinitesimally perturbed circles, i.e. special shapes.

5.1. Special parameter values

The connection between rolling and floating in the limit $\rho\to 0$ enables us to prove

Theorem 6: given an arbitrary sufficiently convex, homogenous floating object Ω, there exists $\varepsilon>0$ *such that it is not monostatic if its density satisfies* $\rho<\varepsilon$.

At the same time, for $\rho\approx1/2$ as well as $d\approx0$ the self-duality of the problem shown in Sections 2 and 3 simplifies the analysis. In these cases we show

Theorem 7: given an arbitrary sufficiently convex, homogenous floating object Ω, there exists $\varepsilon>0$ such that it is not monostatic if its density satisfies $|\rho-1/2|<\varepsilon$.

Theorem 8: given an arbitrary sufficiently convex, homogenous, partially submerged object Ω with centroid pinned at height d, there exists $\varepsilon>0$ such that it is not monostatic if $|d|<\varepsilon$

In each of the theorems, the following subcases are considered:

- *A:* Ω is not weakly monostatic in the limit of rolling (*Thm. 6*), if $\rho=1/2$ (*Thm. 7*) or if $d=0$ (*Thm. 8*).
- *B:* Ω is weakly monostatic but not neutral under the same circumstances
- *C:* Ω is not a circle but floats neutrally under the same circumstances
- *D:* Ω is a circle

Case d) is trivial, since a homogenous circle is not monostatic. So is case a), since an energy function that is not even weakly monostatic cannot become monostatic via infinitesimal perturbations due to continuity arguments. The remaining two cases are discussed below.

Proof of Theorem 6:

Case B

This class is empty [22].

Case C

This class is empty, since all wheels are round, i.e. the circle is the only shape rolling neutrally on a solid surface [25]. □

Proof of Theorem 7:

Case B

We show by contradiction that this class is empty. Assume that the potential energy function $U_{1/2}(\alpha)$ of Ω is weakly monostatic but not constant. Let $0\leq\alpha_1,\alpha_2\leq\pi$ satisfy $U_{1/2}(\alpha_1)=\min U_{1/2}(\alpha)$, $U_{1/2}(\alpha_2)=\max U_{1/2}(\alpha)$; α_1 and α_2 then correspond to equilibria. $U_{1/2}$ is weakly monostatic, thus according to *Observation 1*, there is an α_0 such that $U_{1/2}(\alpha)\geq U_{1/2}(\alpha_0)$

if $\alpha\in(\alpha_0,\alpha_0+\pi)$, and $U_{1/2}(\alpha)\leq U_{1/2}(\alpha_0)$ if $\alpha\in(\alpha_0-\pi,\alpha_0)$. If α_1 is in one of these intervals (mod 2π), then $\alpha_1+\pi$ is contained by the other one. This fact and the π-periodicity of $U(\alpha)$ (shown in Section 2) imply $U_{1/2}(\alpha_1)=U(\alpha_0)$. The same argument also yields $U_{1/2}(\alpha_2)=U(\alpha_0)$, but then $U_{1/2}(\alpha_1)= U_{1/2}(\alpha_2)$, which contradicts the assumption that $U_{1/2}(\alpha)$ is not constant.

Case C

Let $d(\alpha)$ denote the signed height difference between G and the water line w if the orientation of the floating object is α and its density is $\rho=1/2$. Since w cuts Ω to two halves of equal area,

$$d(\alpha)=-d(\alpha+\pi). \tag{29}$$

If ρ is decreased by δ, a narrow band of area δA and height $O(\delta)$ next to w ceases to be immersed and according to (1), the potential energy becomes

$$U_{1/2-\delta}(\alpha)=U_{1/2}(\alpha)-\left(d(\alpha)+O(\delta)\right)\cdot\delta\cdot A=\text{constant}-\delta\cdot A\cdot d(\alpha)+O(\delta^2) \tag{30}$$

If Ω is monostatic for small δ, then $d(\alpha)$ must be at least weakly monostatic. The symmetry (29) of $d(\alpha)$ and *Observation 1* imply the existence of α_0 such that $d(\alpha_0)=d(\alpha_0+\pi)=0$, $d(\alpha)$ is nonnegative in the interval $(\alpha_0,\alpha_0+\pi)$ and nonpositive in $(\alpha_0+\pi,\alpha_0+2\pi)$. Without loss of generality, $\alpha_0=0$ is assumed. We reformulate our statement to a problem of floating with pinned centroid, as follows. For every α, let the water line w be shifted to its parallel copy w_0 going through G≡O. This theoretical shift w increases the immersed area if $\alpha\in(0,\pi)$ and decreases it if $\alpha\in(\pi,2\pi)$. Thus, the resultant immersed area satisfies $A_B(\alpha)\geq A/2$, and $A_B(\alpha)\leq A/2$ in the two respective cases. To provide contradiction and to complete the proof thereby, it is enough to show that

Lemma 1: if $A_B(\alpha)\geq A/2$, and $A_B(\alpha)\leq A/2$ in $\alpha\in(0,\pi)$, and $\alpha\in(\pi,2\pi)$ respectively, then Ω is centrally symmetric.

because the potential energy of a centrally symmetric object is π-periodic, excluding monostatic behaviour for any ρ (cf. the proof of case B).

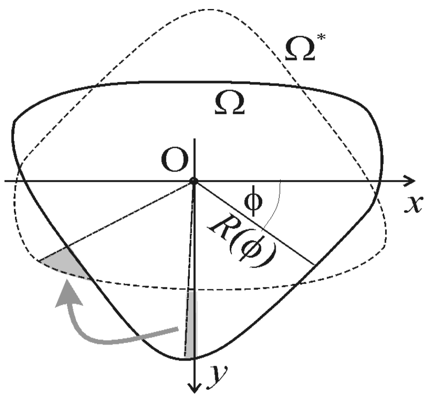

Figure 8*: illustration to the proof of Lemma 1*. Grey arrow shows the initial step of rearranging Ω.

Proof of Lemma 1:

The submerged part is the union of all points, whose polar angle is in the interval (α-π/2,α+π/2), hence we have

$$A_B(\alpha) = \frac{1}{2}\int_{\alpha-\pi/2}^{\alpha+\pi/2} R^2(\phi)d\phi \geq A/2 = \frac{1}{2}\int_{-\pi/2}^{\pi/2} R^2(\phi)d\phi \quad \text{if} \quad 0 \leq \alpha \leq \pi \tag{31}$$

immediately yielding the following two inequalities

$$\int_{\pi/2}^{\pi/2+\alpha} R^2(\phi)d\phi \geq \int_{\pi/2}^{\pi/2+\alpha} R^2(\phi+\pi)d\phi \quad \text{if} \quad 0 \leq \alpha \leq \pi \tag{32}$$

$$\int_{\pi/2-\alpha}^{\pi/2} R^2(\phi)d\phi \geq \int_{\pi/2-\alpha}^{\pi/2} R^2(\phi+\pi)d\phi \quad \text{if} \quad 0 \leq \alpha \leq \pi \tag{33}$$

We now show via a geometric rearrangement of some parts of Ω that it must be centrally symmetric

We assume that $R'(-\pi/2)=0$, i.e. that the tangent of the boundary curve right above O is horizontal. If Ω does not meet this requirement, the following affine transformation

$$(x, y) \mapsto \left(x, y - x\frac{R'\left(-\frac{\pi}{2}\right)}{R\left(-\frac{\pi}{2}\right)} \right) \tag{34}$$

makes the job without changing the position of the centre of gravity or any other important property of Ω.

Let Ω^* be the central reflection of Ω to O! The equation of the boundary of Ω^* is $R(\phi+\pi)$, see the dashed line in Figure 8. We now consider the interval $\phi\in(\pi/2,\pi)$; we gradually remove infinitesimally small radial segments from the ‘excess area’ $\Omega\backslash\Omega^*$ (inside the curve $R(\phi)$ but outside $R(\phi+\pi)$) starting from $\phi=\pi/2$; the removed area is used to fill up the ‘missing area’ $\Omega^*\backslash\Omega$ again from $\pi/2$ towards π. We proceed until the missing area vanishes from the quadrant $\phi\in(\pi/2,\pi)$. The following simple observations together imply that each piece of rearranged area *moves upwards*:

- Due to (32), the polar angle associated with the position of the rearranged pieces always increases (and the procedure is finished before the excess area completely disappears).
- The dashed curve has horizontal tangent at $\phi=\pi/2$, and due to its convexity, the y coordinate montonically decreases along the curve if ϕ is increased in the interval $\phi\in(\pi/2,\pi)$.
- each rearranged segment moves from below the dashed curve to above it.

The same rearrangement procedure can be done in the interval $(0,\pi/2)$, and the rearranged pieces again move upwards. As a result, the centroid of the rearranged object Ω^{**} is above or at the same height as initially:

$$y_{G^{**}}\leq 0 \tag{35}$$

At the same time the new boundary curve $R^{**}(\phi)$ satisfies $R^{**}(\phi)\geq R^{**}(\phi+\pi)$ for every $\phi\in(0,\pi)$, thus the y coordinate of its centroid is

$$y_{G^{**}} = \frac{\frac{1}{3}\int_0^{2\pi} R^{**3}(\phi)\sin\phi d\phi}{A} = \frac{\int_0^{\pi}\left(R^{**3}(\phi)-R^{**3}(\phi+\pi)\right)\sin\phi d\phi}{3A} \geq 0 \tag{36}$$

(35), and (36) imply $y_{G^{**}}=0$. Equality in (36) and (35) occur if $R^{**}(\phi)\equiv R^{**}(\phi+\pi)$ and if no arrangement is needed to eliminate the ‘missing part’, respectively. Hence, $R(\phi)\equiv R(\phi+\pi)$ i.e. the object is centrally symmetric. □

Proof of Theorem 8:

Case B

The proof relies on the π-periodicity of the potential energy function just as in case B of *Theorem 7*.

Case C

If $d=\delta$ is small but positive, the potential energy of the pinned object is

$$U_\delta(\alpha) = U_0(\alpha) - \delta A_B(\alpha) + O(\delta^2) = \text{constant} - \delta A_B(\alpha) + O(\delta^2) \tag{37}$$

where $A_B(\alpha)$ is the area of the immersed part with $d=0$. Hence if Ω is monostatic for small δ, then $A_B(\alpha)$ must be at least weakly monostatic. The identity $A_B(\alpha)+A_B(\alpha+\pi)\equiv A$ and *Observation 1* imply the existence of α_0 such that $A_B(\alpha)\geq A/2$ if $\alpha\in(\alpha_0,\alpha_0+\pi)$ and $A_B(\alpha)\leq A/2$ if $\alpha\in(\alpha_0+\pi,\alpha_0+2\pi)$. We assume $\alpha_0=0$ without loss of generality and we use *Lemma 1* to conclude that the object is centrally symmetric and thus not monostatic. □

5.2. *Almost round shapes*

The problem appears to be difficult for arbitrary shapes and arbitrary values of ρ and d; however the case of nearly round objects is still tractable. Since the circle is neutral (i.e. weakly monostatic), an infinitesimally perturbed circle is a potential candidate for monostatic behaviour. Nevertheless, we show the following two theorems by linearising the effect of the perturbation:

Theorem 9: a floating infinitesimally perturbed circle is not monostatic.

Theorem 10: an infinitesimally perturbed circle floating with pinned centroid is not monostatic.

Proof of Theorem 9: the main arguments of the proof follow those of *Theorem 3*. We consider a perturbed circle in polar coordinate system, cf. (6). We assume that the centre of gravity coincides with the origin implying

$$\varepsilon \int_0^{2\pi} f(x)\sin x\,dx + O(\varepsilon^2) = 0 \tag{38}$$

Assume now that the object (that is: $U_d(\alpha)$) is monostatic. We apply *Observation 1* to $\Psi\equiv U_d$ and assume that $x_0=0$, implying the following inequality:

$$0 < \int_0^{\pi} \left(U(\alpha) - U(\alpha + \pi)\right) \sin \alpha d\alpha = \int_0^{2\pi} U(\alpha) \sin \alpha d\alpha \tag{39}$$

Next, the expression (12) of U is substituted into (39) and a zero term is removed:

$$\begin{aligned} 0 &< \int_0^{2\pi} \left[U_{\rho 0} + \varepsilon \int_{\alpha-\theta}^{\alpha+\theta} f(x)\left(\cos(x-\alpha) - \cos\theta\right) dx + O(\varepsilon^2) \right] \sin \alpha d\alpha = \\ &= \varepsilon \int_0^{2\pi} \int_{\alpha-\theta}^{\alpha+\theta} f(x)\left(\cos(x-\alpha) - \cos\theta\right) dx \sin \alpha d\alpha + O(\varepsilon^2) \end{aligned} \tag{40}$$

Replacing the variable x by y=x-α, and changing the order of integration yield

$$\begin{aligned} 0 &< \varepsilon \int_0^{2\pi} \int_{-\theta}^{\theta} f(y+\alpha)\left(\cos y - \cos\theta\right) \sin \alpha dy d\alpha + O(\varepsilon^2) = \\ &= \varepsilon \int_{-\theta}^{\theta} \int_0^{2\pi} f(y+\alpha)\left(\cos y - \cos\theta\right) \sin \alpha d\alpha dy + O(\varepsilon^2) \end{aligned} \tag{41}$$

We now replace α by β=α+y, exploit the 2π periodicity of the integrated functions, change the order again and express the inner integral in closed form:

$$\begin{aligned} 0 &< \varepsilon \int_{-\theta}^{\theta} \int_{-y}^{2\pi-y} f(\beta)\left(\cos y - \cos\theta\right) \sin(\beta - y) d\alpha dy + O(\varepsilon^2) = \\ &= \varepsilon \int_{-\theta}^{\theta} \int_0^{2\pi} f(\beta)\left(\cos y - \cos\theta\right) \sin(\beta - y) d\beta dy + O(\varepsilon^2) = \\ &= \varepsilon \int_0^{2\pi} \int_{-\theta}^{\theta} f(\beta)\left(\cos y - \cos\theta\right) \sin(\beta - y) dy d\beta + O(\varepsilon^2) = \\ &= \varepsilon \int_0^{2\pi} f(\beta) \int_{-\theta}^{\theta} \left(\cos y - \cos\theta\right) \sin(\beta - y) dy d\beta + O(\varepsilon^2) = \\ &= \left(\theta - \tfrac{1}{2}\sin 2\theta\right) \varepsilon \int_0^{2\pi} f(\beta) \sin \beta d\beta + O(\varepsilon^2) \end{aligned} \tag{42}$$

The result of the last step contradicts (38) unless the O(ε) term is 0 and only the O(ε^2) term is negative. This case means that the perturbation preserves neutral floating in leading order, which is possible (see *Corollary 1* and the references there). However the only infinitesimal perturbations of this type have rotational symmetries, thus they cannot be monostatic. □

Proof of Theorem 10: Thm. 1 implies that this statement is equivalent of *Thm. 9*. □

6. Discussion

In this paper, we demonstrated that pinned and partially submerged planar objects show complex behaviour similarly to freely floating ones. Analogously to Ulam's famous floating body problem, shapes other than circles may float neutrally if their pinned centroids are at the level of liquid surface (d=0), and such shapes probably also exist if $d\neq 0$. The duals of these two problems were also considered, and the non-existence of planar homogenous floating bodies (freely or with pinned centroid) with less than four equilibria was proved in special cases. Nevertheless, the dual problems were not solved in full generality and there are some indications that such shapes might perhaps exist in case of free floating.

One of these is an interesting connection between floating bodies and 'bicycle curves' [26,27]. The latter are those curves, which may represent the front-wheel track of an idealised bicycle moving in either direction with identical rear tracks in the two cases. We showed in Section 2 that

(i) the water lines of a neutrally floating object are chords of constant length $2l$ and

(ii) they are tangential at their midpoints to the water envelope e.

(i) and (ii) are exactly the properties characterizing bicycle curves (where the bicycle is of length l and the rear-wheel track corresponds to the water envelope). However, as also pointed out in Section 2, (i) and (ii) imply only that the centre of buoyancy moves on a circle and not necessarily that this circle is centred at G. If G is off-centre, the floating body corresponding to the closed bicycle curve is *monostatic instead of neutral*. We remark that the problem of closed bicycle curves and the FBP are often claimed in the literature to be equivalent, which goes back to an unproved statement of [12] corrected in a footnote. In fact, all known, closed bicycle curves correspond to the ρ=1/2 case or they have some rotational symmetry, each of which in itself precludes monostatic floating. Bicycle curves without higher order symmetries nevertheless seem to be strong candidates for affirmative solutions of the MFBP if they exist. Another promising possibility is to consider infinitesimal perturbations of nontrivial solutions of the Floating Body Problem. Although the perturbation of circles is not appropriate (cf. Section 5.2), nevertheless the author was unable to prove this for other neutrally floating shapes. Similar techniques may also help solving *Problem 2*. In any case, the final answers to the questions about monostatic objects would either result in new, physically inspired Four vertex-type theorems or in the discovery of a new type of counterintuitive behavior among floating objects.

All questions discussed in this paper naturally extend to three dimensions. The FBP appears to be quite difficult in 3D, and it has not been solved (see however partial negative [11] and positive [10,4] answers). The MFBP is somewhat different. Convex, homogenous 3D objects may have only two equilibria if resting on a solid, horizontal surface [20,21]. In contrast to neutral behaviour, the monostatic property is preserved by small perturbations. Hence, the same shapes are also monostatic for floating if their density ρ is sufficiently small. Nevertheless this is not true for arbitrary density: if $\rho=1/2$, arguments similar to the two dimensional case, imply that every object has *at least six* equilibria (usually: ≥ 2 minima, ≥ 2 maxima and ≥ 2 saddle points of the potential energy). The author's guess is that monostatic objects exist for arbitrary $\rho\neq 1/2$, yet this has not been proved so far.

Finally, we would like to remark that there are countless other situations, in which the possibility of monostatic or neutral behaviour are relevant questions. Let three examples stay here:

Q1. A light, convex, thin-walled, hollow box lies on a solid, horizontal surface and it is partially filled with liquid. Can it behave neutrally if it is not round? Can it be monostatic?

Q2. The same type of box is pinned at its centroid. The questions are the same as in *Q1*.

Q3. If the surface of a convex, homogenous, solid planet is everywhere 'horizontal' (i.e. the surface tangent is perpendicular to the planet's gravitational field), then is it necessarily round? Is it possible that there is only one hilltop and/or one basin on a planet's surface?

It is easy to show that *Q2* is equivalent of the problems of *free floating*. At the same time, *Q1* and *Q3* appear to be unsolved apart from the following special limits: in *Q1*, a box fully filled with liquid behaves like a solid, homogenous object; if however only infinitesimal amount of liquid is placed to the box, then neutral behaviour corresponds to constant Gaussian curvature on its surface, implying roundness according to Liebmann's century-old theorem [28]. As shown by this example, problems like the above ones may not only uncover exciting physical phenomena, but they may provide natural links to classical problems in geometry.

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