A GENERAL MODEL FOR COLLISION-BASED ABRASION PROCESSES

P.L. VÁRKONYI AND G. DOMOKOS,

Department of Mechanics, Materials and Structures

Budapest University of Technology and Economics

Muegyetem rkp. 3, K242, H-1111 Budapest, Hungary

e-mail: vpeter@mit.bme.hu

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Abstract

By relying on existing results about touching probabilities of convex objects, we derive a PDE for general abrasion processes based on collision sequences. Our model shows that collisions with generic abrading objects can be modeled as a linear combination of collisions with three special abrading objects. One of these corresponds to bouncing on a plane, and the Gauss curvature flow, one to abrasion by sand blasting and the uniform normal flow while the third one to abrasion by random impacts of sticks, and the mean curvature flow. Our model lends itself to a natural discretization scheme where the three special collisions correspond to three random events. We discuss some applications to natural processes.

1. Introduction

Ever since the classical paper by Firey (1974) the fundamental mathematical model for the study of pebble abrasion have been geometric PDEs describing *shrinking surfaces*. In natural abrasion processes, the speed of abrasion in the direction of the surface normal **n** is dictated by the abrading environment and the latter may greatly vary. In Firey's model, the speed is proportional to the Gaussian curvature ρ_g, and all convex surfaces converge to the sphere (Firey 1974, Andrews,1999) or to the circle in 2D (Gage, 1984, Gage & Hamilton, 1986). Similar results can be obtained for speed proportional to the mean curvature ρ_m (Brakke, 1978, Huisken 1984). While these results are of fundamental importance and can explain the abrasion of rolling stones, a serious shortcoming of both models is that they do not contain free parameters to account for the large variety of physical abrading environments. Also, they cannot explain the observation that abrasion may lead to flat rather than spherical shapes as in the case of coastal pebbles (Rayleigh, 1944, Wald, 1990, Lorang & Komar, 1990, Domokos et al., 2009b). The former shortcoming sparked the study of many other, more general cases where convex hypersurfaces move with speeds given by homogeneous degree one, concave or convex monotone symmetric functions of the principal curvatures. Chow (1985) for example considered flows by the *k*th root of the *k*-dimensional Gaussian curvature, and the square root of the scalar curvature (Chow, 1987), and Andrews (1994) considered a more general class of such evolution equations. A comprehensive survey of this literature is provided by Andrews (1999, 2003). The main goal of these studies was to investigate limiting geometries (including the sphere) and other, non-spherical homothetic solutions. The existence of the latter would account for the presence of non-spherical pebbles.

While these generalized shrinkage models certainly admit many free parameters, it is not apparent how these parameters (e.g. the power of the curvature) are related to the physical environment; most notably, it is not clear how many free parameters are necessary to describe a general abrasion process.

Here we take a different approach. The correct interpretation of the probabilistic results of Schneider and Weil (2008) leads to a deterministic PDE directly based on the geometry of individual collisions. This PDE appears to be a natural generalization of previous models since it is a *linear combination* of the Gauss curvature flow (Firey, 1974), the mean curvature flow (Huisken 1984) and the uniform normal flow. After rescaling, our PDE contains two free parameters; the speed of abrasion can be written as $p\overline{\rho}_g + q\overline{\rho}_m + 1$, in which $\overline{\rho}_g$, $\overline{\rho}_m$ are scale-free measures of the Gaussian, and mean curvature. The parameters p and q can be directly related to the type of abrading environment. In particular, we point out that in case of collisions with relatively *large* impactors the Gaussian term dominates ($p>>q>>1$), in case of very *thin* impactors the mean curvature term ($p<<1<<q$) and in case of *small* impactors ($p<<q<<1$) the uniform term. One can identify natural processes where some of these special cases are realized with high accuracy. Stones are sometimes eroded by wind-driven sand. From the point of view of sand grains (small objects) this is a Gaussian abrasion process (Firey's model) while the stones themselves deform under the uniform normal flow. On larger scale, the same can be observed on asteroids which are bombarded by small meteorites (Domokos et al. 2009a). We remark that the Gaussian process is also a good approximation of abrasion by rolling.

While it is beyond the scope of this paper to find all homothetic solutions of this equation, we will identify numerically homothetic solutions among shapes of revolution. We also discuss the local stability analysis of the sphere (which is a trivial homothetic solution for any parameter value). It is worth noticing that the 2D version of the same PDE contains only one free parameter and here homothetic solutions are easily identified numerically (Domokos et al., 2009b). Our model captures the essential features of all collision-based abrasion processes while other aspects, e.g. abrasion by friction are not included in these equations. In Section 2, we derive the general PDE, in Section 3, we discuss homothetic solutions. In Section 4, we outline applications as well as a natural discretizations scheme.

2. DERIVATION OF THE EQUATION

Abrasion is often the result of many individual collisions in which small pieces are removed from an object's surface. The (time and space) averaged effect of these collisions can be readily formulated as a partial differential equation (PDE),

$$\dot{\mathbf{x}}^{(A)} = \delta \cdot \frac{\Delta f}{\Delta S} \cdot \mathbf{n}^{(A)} \tag{1}$$

where $\dot{\mathbf{x}}^{(A)}$ is the abrasion speed at a given point A of the surface of the abraded object K in the direction of the unit surface normal $\mathbf{n}_A$; δ is the mean volume removed in one collision event, and $\Delta f/\Delta S$ is the mean intensity of impacts per unit surface area around A. How the latter varies on a surface, depends both on its geometry and on the type of the abrasion process. Alternatively, $\Delta f/\Delta S$ depends both on the geometry of the abraded object and the geometry of the impactors. The classical abrasion model of Firey (1974) is based on the assumption that the abraded object K collides many times to a *plane*. The orientation of K (relative to the impacting plane) is assumed to be a uniform random variable, which implies that the intensity of collisions is proportional to the Gaussian-curvature ρ_g of K (provided that K is convex). For a concave surface, this is only true for those points which span the convex hull of K, other points are never hit by the plane, hence their speed of abrasion is 0. The emerging PDE

$$\dot{\mathbf{x}}^{(A)} = \begin{cases} 0 & \text{inside convex hull} \\ \text{constant} \cdot \rho_g^{(A)} \cdot \mathbf{n}_A & \text{on border of convex hull} \end{cases} \tag{2}$$

makes initially concave surfaces convex. Furthermore, K shrinks, regardless of the initial shape, to a point in finite time (which does not necessarily correspond to finite physical time), and while shrinking, the shape of K converges to a sphere.

The impacting plane can be considered as an idealized model of the ground in case of pebbles; alternatively, it can be thought of as an *infinitely large* impactor. Real stones and other abrading objects however most often occur in the form of granular sets, thus, they are hit by other *finite* impactors (rather than the ground), which affects the distribution of impacts on their surfaces, and thus the emerging shape dynamics. Based on results summarized in a textbook of stochastic geometry by Schneider and Weil (2008), we derive a PDE that captures the essence of this more general process. Similar to Firey's model, it is demonstrated that the impact intensity depends only on the local shape of the abraded surface if we restrict ourselves to interactions of convex objects. Concave shapes are discussed later. For a self-contained derivation of the two-dimensional analogue of this equation, see Domokos et al. (2009b).

If two objects K, and M hit each other in a random fashion, it is straightforward two ask how likely it is that the impact points belong to some given portions A, and B of their respective surfaces. The *touching probabilities* described in Schneider & Weil (2008) provide the answer to this question in case of convex objects. This work does not concern abrasion, nor are any applications outlined in it. We use their formulas, and our equation (1) to obtain the generalized PDE for abrasion processes (as long as they result from collisions). Next, we recall the most relevant theorem of Schneider & Weil (2008), subsequently we will interpret and translate it into the above outlined context of shape dynamics.

Theorem 1 (Thm. 8.5.2 of Schneider and Weil (2008)): Let K, M be convex bodies in R^d, and let A,B Borel sets on their respective boundaries. Let furthermore g denote an arbitrary rigid-body transformation in R^d. Then the natural touching probability of A, and B is

$$p\big(A \cap gB \neq 0 | K, \text{and } gM \text{ touch}\big) = \frac{\sum_{k=0}^{d-1} c_{k,d} \Phi_k(K,A) \Phi_{d-1-k}(M,B)}{\sum_{k=0}^{d-1} c_{k,d} \Phi_k(K,\text{bd}K) \Phi_{d-1-k}(M,\text{bd}M)} = \tag{3}$$

In (3), $c_{k,d}$ are constants; bd$K$ denotes the whole boundary of K, and the denominator of the right side of is just a normalization factor. $\Phi_k(K,A)$ means the k^{th} curvature measure of bd$K$ on A; for smooth bodies, this is the integral of the $(d\text{-}1\text{-}k)^{\text{th}}$ normalized elementary symmetric function ($\rho_{d\text{-}1\text{-}k}$) of the principal curvatures (κ_i) of bd$K$ over A:

$$\Phi_k(K,A) = \int_A \rho_{d-1-k}\, dS \overset{def}{=} \begin{cases} \binom{d-1}{k}^{-1} \cdot \int_A \left(\sum_{j_1<j_2<\ldots<j_{d-1-k}} \kappa_{j_1}\kappa_{j_2}\cdot\ldots\cdot\kappa_{j_{d-1-k}} \right) dS & \text{if } 0 \le k \le d-2 \\ \int_A dS & \text{if } k = d-1 \end{cases} \tag{4}$$

Hence, in case of 3-dimensional objects, Φ_0, Φ_1, and Φ_2 are the surface area, the integral of the mean curvature, and the integral of the Gaussian curvature over A, respectively.

The *natural* touching probability in the theorem is originally defined in a highly technical way. We use the following equivalent definition of a 'natural' random touching event:

- one of the two objects (say K) is kept fixed, the other one moves along an oriented straight line without rotation until it touches K for the first time.
- The motion should satisfy the 'touching condition', i.e. M is required to hit K while moving along its trajectory.
- the parameters of this process are all *uniform* random variables drawn from the finite-sized domain of the parameter space that satisfies the touching condition. These parameters are a uniform random rotation (defining the orientation of M), and a uniform, random, oriented line (defining the trajectory of its centroid).

Hence, the theorem is suitable to obtain distributions of impacts in random collisions.

For the purpose of the current paper, let K be the convex, 3-dimensional object whose abrasion is considered, and A an infinitesimal piece of area ΔS on its surface, where the mean and Gaussian curvatures of bd$K$ are $\rho_m^{(A)}$, and $\rho_g^{(A)}$, respectively. Let M be another convex object, which represents the environment of K, B is the entire boundary of M (B=bd$M$) $\rho_m^{(M)}$, and $\rho_g^{(M)}$ denote mean- and Gaussian curvature functions on bd$M$. With this choice, *Theorem 1* yields the probability that A receives impact in a random collision between K and M:

$$p(A \text{ receives impact}) \sim \sum_{k=0}^{2} c_{k,3}\Phi_k(K,A)\Phi_{2-k}(M,\text{bd}M) \tag{5}$$

In the above formula,

$\Phi_0(K,A)=\rho_g^{(A)}\Delta S$,

$\Phi_1(K,A)=\rho_m^{(A)}\Delta S$,

$\Phi_2(K,A)=\Delta S$ (6)

$$\Phi_0(M,\text{bd}M) = \int_{\text{bd}M} \rho_g^{(M)} dS = 4\pi \tag{7}$$

(8)

(9)

$$\Phi_1(M,\text{bd}M) = \int_{\text{bd}M} \rho_m^{(M)} dS \overset{def}{=} I^{(M)} \tag{10}$$

(11)

$$\Phi_2(M,\text{bd}M) = \int_{\text{bd}M} dS \overset{def}{=} S^{(M)}$$

where $S^{(M)}$ is the surface area, and $I^{(M)}$ is the mean curvature integral of M; furthermore, the integral of the Gaussian curvature is a topological invariant, which equals 4π for convex, 3D objects. We substitute the above equations as well as $c_{0,3}=1$, $c_{1,3}=2$, $c_{2,3}=1$ (for more details, the reader is advised to consult Schneider & Weil (2008)) into (5):

$$p(A \text{ receives impact}) \sim \rho_g^{(A)} \Delta S \cdot S^{(M)} + 2\rho_m^{(A)} \Delta S \cdot I^{(M)} + \Delta S \cdot 4\pi \tag{12}$$

According to (1), the abrasion speed of *K* at *A* is proportional to

$$\dot{\mathbf{x}}_A \sim \frac{p(A \text{ receives impact})}{\Delta S} \cdot \mathbf{n}_\alpha \tag{13}$$

By scaling the speed, the right side of (1) is divided by 4π, and the ~ sign is replaced by the = sign, so our final PDE is

$$\dot{\mathbf{x}}_A = \left[\frac{S^{(M)}}{4\pi} \rho_g^{(A)} + \frac{I^{(M)}}{2\pi} \rho_m^{(A)} + 1 \right] \mathbf{n}_A . \tag{14}$$

As we can observe, the abrasion speed is a linear combination of three terms, a constant one, and two others proportional to the mean and Gaussian curvatures, respectively. The shape dynamics described by (14) is invariant to scaling of sizes, this is, however, not reflected by the parameters $S^{(M)}$, and $I^{(M)}$. To enhance further analysis, we use the dimensionless form:

$$\dot{\mathbf{x}}_A = \left[\frac{S^{(M)}}{S^{(K)}} \cdot \frac{S^{(K)}}{4\pi} \rho_g^{(A)} + 2\frac{I^{(M)}}{I^{(K)}} \cdot \frac{I^{(K)}}{4\pi} \rho_m^{(A)} + 1 \right] \mathbf{n}_A \stackrel{def}{=} \left[\bar{S} \bar{\rho}_g^{(A)} + 2\bar{I} \bar{\rho}_m^{(A)} + 1 \right] \mathbf{n}_A \tag{15}$$

where $\bar{S}, \bar{I}$ are the relative surface area and mean curvature integral of *M* compared to *K*, hence $\bar{S} = \bar{I} = 1$ if *M*, and *K* are identical. $\bar{\rho}_g$, and $\bar{\rho}_m$ are scale-free curvatures, which are size-invariant, e.g. they are 1 on the surface of a sphere of arbitrary radius. By adopting the notation of the Introdcution we have $p = \bar{S}, q = 2\bar{I}$. This model includes Firey's equation if the environment consists of big particles (yielding $p>>q>>1$); the constant term dominates if *M* is much smaller than *K* (hence $p<q<1$); finally, the mean curvature term may also be dominant, if *M* is long and narrow ($p<<1<<q$), although this limit is physically less relevant due to the fragility of such shapes in physical processes.

3. HOMOTHETIC SOLUTIONS

From the point of view of shape dynamics, it would be highly interesting to know where typical initial shapes evolve to at certain parameter values. A solution of (15) might diverge (e.g. with unboundedly growing flatness) or it may converge to a homothetic solution, so it is useful to learn about the latter and their stability.

As already mentioned, the behavior of the equation is well understood in case of abrasion purely by Gaussian curvature or by mean curvature. In case of purely uniform abrasion, homothetic solutions have been discussed by Pegden (2009), most of them are, however, unstable. Typical initial shapes may converge to an arbitrary tetrahedron (which is the only attractive homothetic solution), alternatively they may become either infinitely flat, or infinitely elongated. The fate of an initial shape depends on the number of touching points of its largest inscribed ball, which is 4, 3, or 2 in a generic situation. For more details, see Domokos et al (2009a,b).

If there are several non-vanishing terms in (15), little is known about the emerging dynamics. In fact, almost all analytic results concerning curvature dependent interface motion are restricted to homogenous (and in most cases only degree 1) functions of the principle curvatures (e.g. Andrews, 1999, Andrews 2000, Chow 1985,1987; Huisken 1984), and our equation is not of this type. Here, we show numerical results about homothetic solutions restricted to bodies of revolution.

Let $R(\phi)$ ($0 \le \phi \le \pi$) denote the equation of the generating curve of an object of revolution in polar coordinates. This shape is a homothetic solution (that contracts to the origin of the coordinate system) if R is proportional to the abrasion speed in radial direction, i.e. by (14) if

$$R(\phi) = c\left[\frac{S^{(M)}}{4\pi}\rho_g(\phi) + \frac{I^{(M)}}{2\pi}\rho_m(\phi) + 1\right]\sqrt{1 + \frac{R'^2(\phi)}{R^2(\phi)}} \tag{16}$$

with some constant c. (Notice that the square root term transforms abrasion speed in the direction of the surface normal to radial speed.) ρ_G, and ρ_m can be expressed as functions of R, R', and R'' (not shown), and (16) can be solved as a boundary value problem with boundary conditions $R(0)=1$ (which normalizes the size of the solutions), and $R'(0)=R'(\pi)=0$ (which guarantee smoothness at the poles of the object). For each triple c, $S^{(M)}$, $I^{(M)}$ solving the boundary value problem, the surface area, and the mean curvature integral of the corresponding object were calculated, and $\bar{S}, \bar{I}$ were determined accordingly (Figure 1).

The above described analysis uncovered the trivial homothetic solution L_0 (the sphere), and branches of nontrivial ones L_i (i=2, 3, 4,…) which are labeled by the number i of curvature maxima in the silhouette of the corresponding shapes. Among these, L_2 is especially interesting, since it consists of elongated, as well as flat surfaces (on the two sides of L_0), and the latter resemble flat beach pebbles.

Our analysis does not provide information about the stability of these solutions. Nevertheless, Andrews (1996) investigated the closely related equation

$$\dot{\mathbf{x}}_A = \left[\rho_g^{(A)}\right]^p \mathbf{n}_A \tag{17}$$

and conjectured that the sphere is globally stable if the exponent p exceeds 1/4, and unstable if $p<1/4$. (This has been proved only for the specific values $p=1/2$ (Chow, 1985) and 1. If $p=1/4$, it is known that all ellipsoids are homothetic solutions, and the sphere is neutral, see Andrews (1996)). In order to connect these results to our equation, we linearize (17) about the sphere. Without loss of generality, we consider a perturbed unit sphere whose principal curvatures are $1+\varepsilon\kappa_i$, i=1,2. Then, (17) becomes

$$\dot{\mathbf{x}}_A = \left[(1+\varepsilon\kappa_1)(1+\varepsilon\kappa_2)\right]^p \mathbf{n}_A = \left[1 + p\varepsilon(\kappa_1+\kappa_2) + O(\varepsilon^2)\right]\mathbf{n}_A \tag{18}$$

Similarly, the linearization of (14) is

$$\begin{aligned}\dot{\mathbf{x}}_A &= \left[\frac{S^{(M)}}{4\pi}(1+\varepsilon\kappa_1)(1+\varepsilon\kappa_2) + \frac{I^{(M)}}{2\pi}\left(1 + \frac{\varepsilon\kappa_1 + \varepsilon\kappa}{2}{}_2\right) + 1\right]\mathbf{n}_A = \\ &= \left[1 + \frac{S^{(M)}}{4\pi} + \frac{I^{(M)}}{2\pi} + \left(\frac{S^{(M)}}{4\pi} + \frac{I^{(M)}}{4\pi}\right)\cdot\varepsilon(\kappa_1+\kappa_2) + O(\varepsilon^2)\right]\mathbf{n}_A\end{aligned} \tag{19}$$

The two linearized equations are equivalent, if the relative weights of the constant, and the curvature-dependent term are equal, hence if

$$\frac{\frac{S^{(M)}}{4\pi}+\frac{I^{(M)}}{4\pi}}{1+\frac{S^{(M)}}{4\pi}+\frac{I^{(M)}}{2\pi}}=p \tag{20}$$

Since the object in question is a perturbed unit sphere, $I^{(K)}=S^{(K)}=4\pi+O(\varepsilon)$. Thus, by definition of $\bar{S},\bar{I}$, (20) becomes

$$\frac{\bar{S}+\bar{I}}{1+\bar{S}+2\bar{I}}=p\,, \tag{21}$$

and the case $p > 1/4$ corresponds to

$$3\bar{S}+2\bar{I}>1\,. \tag{22}$$

Thus, if the conjecture of Andrews is correct, then the sphere is a *locally* attracting shape in (15) if (22) holds, and it is unstable in the opposite case. In fact, these two regions lie on the two sides of the L_2 branch in Figure 1.

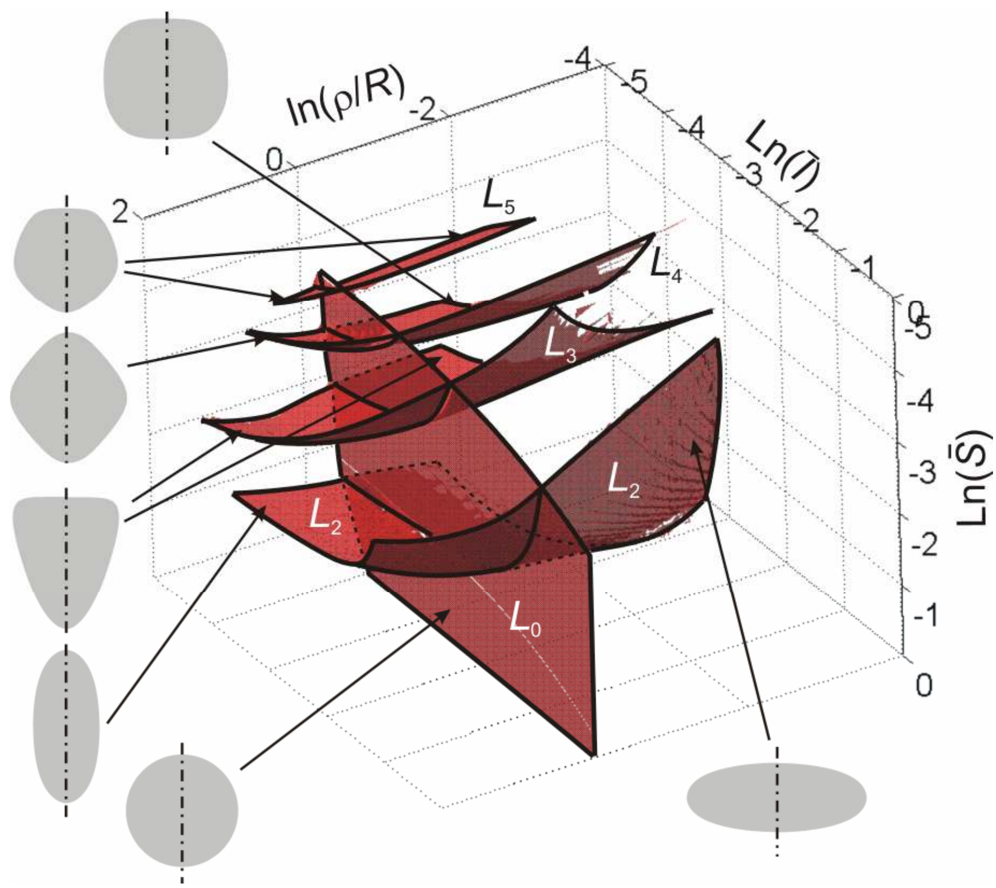

Figure 1: Numerically obtained bifurcation diagram of homothetic solutions in logarithmically scaled parameter space. Edges of the surfaces are highlighted for better visibility. $\bar{I}$ and $\bar{S}$ are the parameters of equation (15); the solutions are parametrized by ρ*R*, where ρ is the curvature of the contour at its bottom, and *R* is the distance of this point from the point where the homothetic solution contracts to. Some corresponding contours are also shown. The diagram shows a series of nontrivial families L_i of solutions bifurcating from the trivial solution L_0 (sphere; ρ*R*=1). If *i* is odd, the two halves on the two sides of L_0 contain the same set of solutions. In the parameter regimes $\bar{S}\geq 1$, $\bar{I}\geq 1$ (not shown in the plot) nothing but the trivial solution was identified.

4. APPLICATIONS, DISCRETIZATIONS AND CONCLUSIONS

4.1 Applications and limitations

Equation (15) contains three terms two of which can be observed in Nature as independent abrasion processes under realistic conditions. The first term is related to the Gaussian curvature and, as already noted, corresponds (approximately) to abrasion by rolling. This process can be also understood as a sequence of collisions with *infinitely large* (planar) impactors, hence the coefficient $\overline{S}$ will dominate the equation. As first pointed out by Firey (1974) and later proven by Andrews (1999) the limiting geometry is a sphere.

The third (constant) term can be observed in abrasion processes with very small impactors. On geological scale, this can be considered as an idealized model of the formation of *ventifacts*, i.e. stones shaped by wind-blown sand (Bourke, and Viles, 2007). Several aspects of ventifact formation (most notably the anisotropy of wind directions) are neglected by our model, nevertheless many of the observable ventifact shapes strongly resemble the limit configurations of Section 3 (cf Fig 2G). Currently, we are studying this abrasion process in more detail. On larger scale, an apparently similar process governs the abrasion of asteroids, via collisions to minor particles (Domokos et al. 2009a.) Figure 2 shows a simulation as well as an asteroid shape reconstructed by photometric imaging.

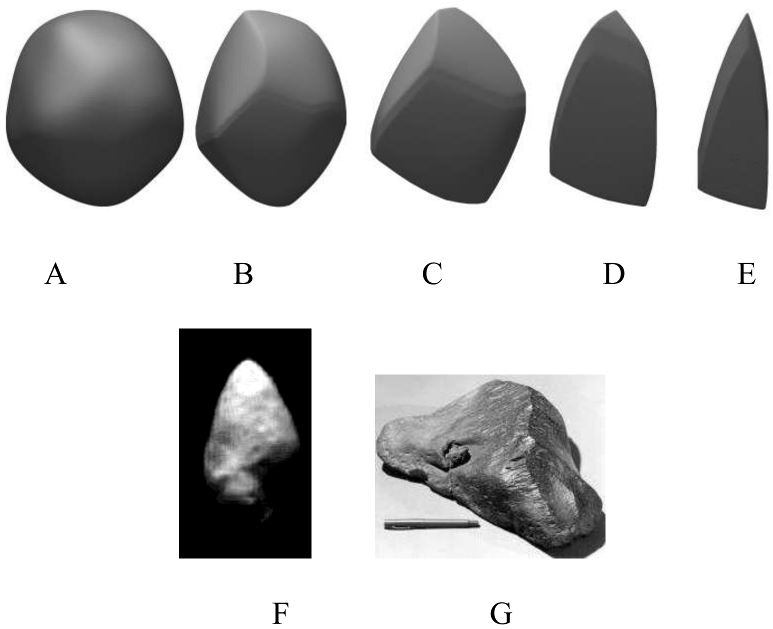

Figure 2 A-E: Simulation of (15) with $\overline{S} = \overline{I} = 0$ (*p*=*q*=0), i.e. uniform normal flow. Random, smooth initial shape (A) is being abraded into an object with distinct sharp edges and flat areas (B-E) (source: Domokos et al., 2009a). Observe qualitative similarity of D to F: observed asteroid shape Annefrank 5545 (source: http://stardust.jpl.nasa.gov), and G: ventifact from Mojave desert, California (source: Greeley et a., 2002; a pen is also shown for comparision of sizes) both with sharp edges.

Equation (15) represents the abrasion due to one particular object M, however, it naturally extends to the case of many different abraders: in the latter case, the parameters p, q represent appropriate averages characterizing the abrading environment. This is relevant in a geological setting: it shows that interacting particles abraded by mutual collisions exhibit different types of abrasion according to their sizes. The interactions of sand particles and bigger stones has already been pointed out as an example. A similar effect must be present among pebbles, and our model suggests that in an assembly with mixed sizes, smaller pebbles are more likely to become round, while big ones tend to evolve into different shapes. More work is needed to learn about the shapes predicted by our model, and its relation to pebble abrasion.

As already mentioned, PDE (15) is only valid for convex shapes, however, the initial shapes of abrading objects are very often non-convex (concave). In such a situation, concave parts of the object are less exposed to (in case of small abraders) or completely protected from (in case of big abraders) the impacts, resulting in lower abrasion speed and in most cases the elimination of the concavity. While it is certainly not true that every single shape becomes convex eventually, we think that that this is the most common case. Moreover, the opposite requires special initial shapes that are less likely to be formed in natural processes (such as fragmentation).

Nevertheless, there are other, serious limitations of the model: on one hand, the global geometry of K may influence the abrasion speed at a given point (e.g. if K and M are allowed to rotate, or if gravitational forces modify the trajectories of the colliding objects); on the other hand, it is often not clear whether sequences of random collisions are an adequate model for all abrasion processes, alternatively, it is not always clear how the relative probabilities associated with the three terms should be derived from the physical process. For example, beach pebbles are subject to relatively small impacts by the wave current; these are insufficient to trigger uniform random collisions. Instead, pebbles mostly slip or roll on each other, and in that case, the intensity of impacts on given surface portions is influenced by the global shape of the abraded object (e.g. flat pebbles mostly slip on their flat parts, which helps maintaining their flatness). Another example is the effect of friction-related abrasion in sliding contacts; this could be approximated in our model by the uniform flow, however, this approach is heuristic. Similarly, there is a heuristic analogy between the effect of abrasion by rolling, and the first term in our equation.

4.2 A natural discretization scheme

Equation (15) lends itself naturally to a random discretizations scheme; the three terms correspond to three random events with total probability one. In fact, this is a new interpretation of equation (12) for the probability of collisions by assigning direct physical meaning to the three additive terms. The probability associated with the individual events corresponds to the relative weights of the three terms p, q, 1. The surface is now approximated by a convex polyhedron and the three events are the following:

(A) The impactor is large and flat; collision occurs between a face of M and a vertex of K. Impact location on K is selected randomly based on solid angles of the surface normal; in case of uniform radial intensity uniform distribution is assumed. Sharp vertices are selected with high probability. In this case a vertex of K is chopped of and replaced by a small face, normal to the randomly selected direction. A random variable with lognormal distribution determines the volume of the chopped particle.

(B) The impactor is large and thin; collision occurs between an edge of *M* and an edge of *K*. Impact location on *K* is selected randomly based on total product of edge-length and edge-angle; in case of uniform radial intensity uniform distribution is assumed. Sharp and long edges are selected with high probability. In this case an edge of *K* is chopped of and replaced by a small, thin face, normal to the randomly selected direction.

(C) The impactor is much smaller than the object; collision occurs between a vertex of *M* and a face of *K*. Impact location on *K* is selected randomly based on surface area; in case of uniform radial intensity uniform distribution is assumed. Large faces are selected with high probability. In this case a face of *K* retreats parallel to itself.

The above scheme can be either regarded as the "sequentially updated" discretizations of the PDE (15) or, alternatively, as a direct discrete model of the physical process.

4.3 Conclusions

We derived a general abrasion model in the form of a PDE, based on individual collisions. Using the results of Schneider and Weil (2008) we showed that collisions with impactors of *arbitrary geometry* can be understood as the linear combination of collisions with impactors with *special geometry*. In particular, three special types of impactors have to be considered: the infinitely large one, causing abrasion proportional to the Gaussian curvature, the infinitely small one, causing abrasion proportional to the surface area and the infinitely thin on, causing abrasion proportional to the mean curvature. We pointed out that the first two types appear in Nature as separate processes as well: infinitely large impactors represent an approximation to abrasion by rolling while infinitely small impactors represent an approximation to abrasion of stones by sand blasting or abrasion of asteroids by collisions with small meterorites. We also presented a natural discretization scheme, which regards the mentioned three special cases as independent random events. We made some modest steps towards the understanding of our model's behavior. Though 'worn' is often considered as a synonym of 'rounded', and deviations from round shapes are sometimes just imperfections (Durian et al. 2006), nevertheless our preliminary results indicate that spontaneous symmetry braking and the development of nontrivial shapes might inherently be present in even the most idealized abrasion processes.

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