Stabilization of rigid bodies with frictional supports against dynamic perturbations

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***Abstract*• Quasi-static robots supported by multiple contacts and Coulomb friction may be overturned by unwanted, arbitrarily small vibrations, under certain geometric conditions. The exact criteria of passive local dynamic stability for perfectly rigid objects (a natural idealization of robots) are currently unknown. In the present paper, we propose a minimalistic, robust, model-independent active stabilization scheme of planar objects via the manipulation of friction forces by wheels and brakes. The potential extensions of the new method to global stabilization, and to three dimensions are also discussed.**

I. Introduction

An essential goal of designing the motion of quasi-static robots is to ensure their balance and stability at all times. For example, the safety of a robot walking on a complex terrain relies on the real-time planning of each step, which is only possible if simple equilibrium and stability conditions are available.

The nontrivial nature of equilibrium conditions originates from the fact that the contact and frictional forces at multiple contact points are not determined uniquely by the equations of motion. An object is in equilibrium if the equations of motion have *static* solution(s) consistent with the contact and friction model. Even in this case, the equations of an object subject to Coulomb friction may have static and nonstatic consistent solutions, simultaneously [1]. In such situations, the equilibrium is called *ambigous*, and it cannot be considered as stable in any sense. On the contrary, if the consistent solutions are all static, then an object with multiple (2 in 2D; 3 in 3D) contact points stays completely immobile in the presence of small perturbing forces, provided that the contact forces are strictly positive and none of them is at the boundary of the friction cone. We refer to this property as *static stability*. The geometric characterization of consistent equilibria and unambigousness has been worked out recently [2].

Static stability is essential in all applications, but it is often insufficient. A robot (modeled as a rigid body) is often subject to small vibrations or other dynamic perturbations, which may break the contacts and cause microscopic motion. The motion is often damped out by the energy absorption associated with collisions and sliding friction; in this case, the object becomes immobile in a small neighborhood of the initial configuration. Nevertheless, the energy loss may be compensated by decreasing potential energy (e.g. if the object moves downhill), and the initial motion may accelerate leading to macroscopic motion and instability. The conditions of "*local dynamic stability*" are unknown, because even the microscopic motion of the object is inherently discontinuous (e.g. bouncing) and nonlinear (because the equations of motion change whenever the number and type (sliding/sticking) of contact points changes during motion). Thus linear stability analysis may not be used. Some sufficient stability condition for planar objects have been worked out recently [3,4], but these are highly limited, and strongly depend on the model of frictional impact used for the analysis. In three dimensions, even sufficient conditions are unavailable.

The gap between the demand of practical engineering and currently available solutions naturally calls for stabilization against dynamic perturbations by some type of energy absorber or by active control. The most straightforward tools are shock absorbers, which indeed help preserving the contacts in the case of perturbations. However, if bouncing motion has already been initiated, absorbers often prove unsuccesful: even objects with perfectly plastic impacts may exhibit exponentially growing bouncing motion on slopes [4]. In this paper, we propose a simple active stabilization strategy of planar objects with 2 contact points, based on monitoring the intensity of contact forces, and controlling *frictional forces* via wheels and adjustable brakes. The clue to this scheme is the understanding of frictionless dynamics, which is discussed in Section II. The full control strategy is presented in Section III, and the Section IV is devoted to extensions to global stabilization and to three dimensions. The paper is complemented by an Appendix containing some technical details of the proofs.

II. Frictionless dynamics

We consider a planar, frictionless object of unit mass and radius of inertia r with two contact points, subject to gravitational forces with gravitational constant g=1, initially resting in contact to an arbitrary smooth terrain without overhang (Fig. 1A). It is assumed that collisions are not perfectly elastic, i.e. the coefficient of restitution is $0\le e<1$. The object is perturbed at t=0 by a small initial displacement or velocity increasing its mechanical energy $E(t)$ from 0 to ε^2 ($\varepsilon \ll 1$).

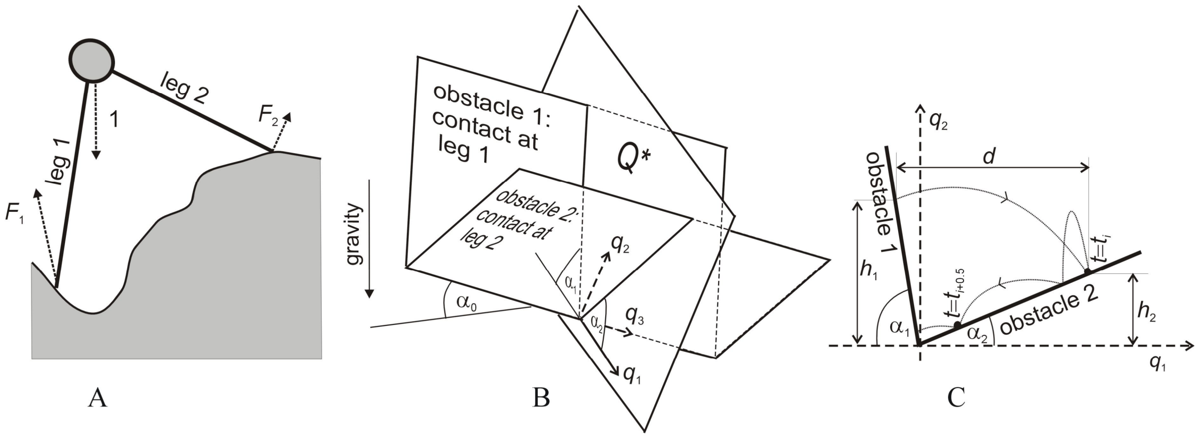

Fig. 1(A): rigid body with 2 contact points on a smooth terrain without overhang. (B): representation of the infinitesimal neighborhood of the unperturbed configuration in the three-dimensional configuration space Q. Configurations are represented by a Cartesian coordinate system (q_i), which is defined uniquely by the following three properties: the origin corresponds to the unperturbed configuration; axis q_3 coincides with the bottom of the wedge; and q_1 is perpendicular to the gravitational acceleration. The geometry of the obstacles is determined by α_0 denoting the "slope angle" of the q_3 axis, as well as α_1 and α_2, i.e. the angles between the obstacles and the q_1 axis. (C): illustration of the microscopic dynamics in the reduced configuration space Q^*.

To enhance further discussion, we characterize the configurations of the object by the three vector $\mathbf{q}$=[x,y, $r\theta$]$\in$Q where x and y are the coordinates of its center of mass and θ is a rotation angle with respect to the initial configuration. The initial configuration is $\mathbf{q}$=0. The configuration space is parametrized by the coordinates q_i, i=1,2,3. The regions of the configuration space Q accessible to the object are bounded by 'obstacles' corresponding to contact between the object and the underlying terrain. In the neighborhood of the origin, the obstacles form a V-shaped wedge (Fig. 1B).

It is convenient to replace the frictionless dynamics of the object by the corresponding dynamics of an imaginary point mass in Q because:

- gravity in physical space causes uniform acceleration $\ddot{q} = [0 -1\ 0]$ in Q

- the kinetic energy of the physical object is $\dot{q}^{\mathrm{T}}\dot{q}/2$ i.e. it takes the form of the kinetic energy of a point mass of unit mass in Q.

- frictionless contact forces in physical space correspond to acceleration of the point perpendicularly to the associated obstacle. Moreover, frictionless impacts in physical space correspond to frictionless bouncing of the imaginary point on an obstacle with the same coefficient of restitution, and sliding appears as frictionless sliding in Q [4].

A. Dynamics over a horizontal line

First, we analyze the case of a horizontal terrain implying α_0=0 in Fig. 1B. We assume that resting on the line is a consistent solution of the equations of motion with strictly positive contact forces implying $0<\alpha_1,\alpha_2<\pi/2$ in Fig. 1B,C. This system is invariant to q_3, and the velocity component $\dot{q}_3$ is constant during motion. Hence it is enough to consider the dynamics projected to the planar section $Q^*=[q_1\ q_2\ 0]$ of Q (Fig. 1B,C). The point $\mathbf{q}^*$=[0 0] is the lowest point of the accessible region in Q^*. Both sliding and resting in physical space correspond to static equilibrium at $\mathbf{q}^*$=0 in Q^*. Let $E^*(t)$ be the apparent mechanical energy of the system projected to Q^* (i.e. the mechanical energy of the system, minus the constant kinetic energy associated to the velocity component $\dot{q}_3$). Then E^*=0 iff $\dot{q}^* = q^* = 0$ and E^*>0 in any other case. We show

Proposition 1: If the underlying terrain is a horizontal line, and resting on the line is a consistent solution of the equations of motion with strictly positive contact forces then the perturbed object reestablishes both contacts within time c_1e for some positive scalar c_1, and performs uniform sliding motion thereafter.

Sketch of the proof: let t_0=0. We identify a series of special impact events at times t_i (i=1,2,…) such that

(i) After event i, the mechanical energy becomes $E^*(t_i^+) \le c_2 E^*(t_{i-1}^+)$ with some constant $0<c_2<1$ and

(ii) if $i\geq 0$, the time difference t_{i+1}-t_i does not exceed $c_3 (E^*(t_i^+))^{1/2}$

Property (i) implies that $E^*(t_i^+) \le c_2^i\varepsilon^2$ hence the series $E^*(t_i^+)$, i=1,2,… converges to zero. At the same time, property (ii) implies that $t_i \le c_3\varepsilon(2+c_2^{1/2}+c_2^{2/2}+\ldots\ c_2^{2(i-1)2}) \le c_3(1+1/(1-c_2^{1/2}))\cdot\varepsilon$ for any i. Thus, we have found a constant c_1 such that E^* converges to zero before $t= c_1\varepsilon$. Since $E_i^*(t)$ is nonnegative and non-increasing, it is exactly zero after this time which completes the proof. The missing technical details of the proof are presented in Appendix V.A.

B. Dynamics over arbitrary terrain

The main difficulty in generalizing *Proposition 1* to this case is that the dynamics projected to Q^* and that of q_3 become coupled unless the underlying terrain is a perfectly flat slope. Nevertheless the microscopic motion in the inifinitesimal neighborhood of the origin is perfectly analogous to motion over a horizontal terrain, yielding

Proposition 2: let $\boldsymbol{q}=0$ *be a configuration of a rigid body with 2 frictionless contact points on a smooth terrain without overhang, such that the object released from this configuration with zero velocity would initially slide on both legs exhibiting strictly positive contact forces. We consider a microscopic dynamic perturbation as in Proposition 1. Then, both contacts are reestablished after infinitesimally short time, and the object continues with (typically nonuniform) sliding.*

Proof of Proposition 2: if the underlying terrain is a flat slope, then the dynamics preserves its translation-invariance, and the proof of *Proposition 1* applies with one single difference: the bottom of the wedge-shaped obstacle in Q has a slope ($\alpha_0 \neq 0$ in Fig. 1B). Thus, the object exhibits uniform downhill *acceleration* of intensity $\sin\alpha_0$ along the longitudinal axis of the wedge. The dynamics projected to a transversal plane Q^* remains the same as described in *Proposition 1* except that the projection to Q^* reduces the gravitational acceleration from 1 to $\cos\alpha_0$. Hence, both contacts are reestablished, within infinitesimal time, but the sliding motion is now nonuniform.

With arbitrary terrain, the problem is no more translation invariant, i.e. the wedge in Q becomes curved. Nevertheless the first-order approximation of the obstacle geometry in the infinitesimal neighborhood of the origin is a straight V-shaped wedge with a sloppy bottom as described above. Using the approximate obstacle geometry is acceptable as long as the configuration of the object is near the origin and its velocity is small. Both are true for the microscopic motion, hence the object exhibits the same behavior as on a flat slope. Overhang in the terrain is excluded to ensure $\alpha_i<\pi/2$ in configuration space. This restriction could be relaxed at the price of a longer proof with more technical details.

III. The Stabilization Scheme

It has been shown on Section II. that microscopic, frictionless bouncing motion always terminates within infinitesimal time and gives rise to pure sliding (or static equilibrium). Despite the fact that friction is the source of additional energy absorption, frictional objects may exhibit persistent bouncing motion with any coefficient of restitution [4]. Hence, frictionless motion is not only technically easier to analyze, but it has fundamental advantages from the point of stabilizing rigid bodies. Our stabilization scheme is based on this result and it can be summarized as follows: *make the motion frictionless until bouncing stops, then apply the brakes gently enough to preserve contacts but strong enough to stop the object*.

Formally, let μ_i (i=1,2) denote the Coulomb friction coefficients in the neighborhood of the contacts. We propose to observe the current state of the object by measuring the intensity of the time-dependent contact forces $F_i(t)$ acting on the wheels. Frictionless motion can be realized by small and light wheels; these are equipped by continuously adjustable brakes. We assume that the state of each brake is characterized by continuous variables $\nu_i(t)$, such that the brake slips if the tangential to normal contact force ratio exceeds $\nu_i(t)$. With other words, the brakes maximize the effective friction coefficient. The stabilization strategy is described by

$$\begin{cases} \nu_i(t)=0 & \text{if} \quad S=0 \\ \dfrac{d\nu_i(t)}{dt}=c\cdot\max\left(0,\dfrac{F_j(t)}{F_i(t)}-d\right) & \text{if} \quad S=1 \end{cases} \qquad (1)$$

where i=1 and j=2 or vice versa; c is a sufficiently big and d is a sufficiently small constant; $S=\mathrm{sign}(F_1(t)\cdot F_2(t))$ is a state indicator of the object. This strategy leads to stabilization as stated by

Theorem 1: let a rigid object with two contacts on arbitrary terrain be equipped by idealized wheels and adjustable brakes such that

(i) The brakes are initially active with $\nu_i>\mu_i$

(ii) The initial configuration is statically stable equilibrium (as defined in the Introduction).

(iii) with brakes released, the object initially exhibits positive contact forces

(iv) the brakes are adjusted according to (1)

then the object reaches static equilibrium after an infinitesimal, dynamical perturbation.

Sketch of the proof: During microscopic motion, the velocity of the sliding object is small and its configuration is in a small neighborhood of the unperturbed configuration. Thus, the dynamics is considered as translation invariant similarly to the approximation of *Proposition 2.*

We make distinction between five modes of motion: static equilibrium; sliding on both legs 'uphill' or 'downhill'; motion with less than 2 contact points in 'uphill' or 'downhill' direction. Downhill and uphill motion are distinguished by the sign of $\dot{q}_3$ in configuration space such that downhill corresponds to the direction in which the object slides if the brakes are released in the initial configuration. If the initial configuration happens to be a frictionless equilibrium, the distinction of uphill and downhill does not make sense, yet the proof is analogous to the general case discussed below.

The proof relies on constructing a transition graph of the five modes (Fig. 2) based on the following considerations. Motion with less than 2 contacts is frictionless by (1) and it exhibits $\ddot{q}_3$=constant such that transition from uphill to downhill is possible but the opposite is not. The frictionless motion terminates within infinitesimal time by *Proposition 2* giving rise to sliding in the same direction. Uphill sliding would stop within infinitesimal time even without friction. The application of the brakes preserves the contacts as shown in Appendix V.B., and sliding stops even sooner than without friction. If uphill sliding stops, the brakes may or may not be active enough to keep the object balanced. In the latter case, the object may continue with downhill sliding; or it may break a contact. If the double contact is broken, downhill frictionless motion follows. In every possible case, *downhill sliding* is reached within infinitesimal time.

In the course of downhill sliding, the controller preserves both contacts, i.e. transition to motion with less than two contacts is impossible (see proof in Appendix V.B.). It is also demonstrated (Appendix V.C.) that the brakes are activated strongly enough to overcome gravity and stop the object within O(c^{-1}) (i.e. very short) time. When the object has stopped, the brakes are obviously active enough to keep it balanced.

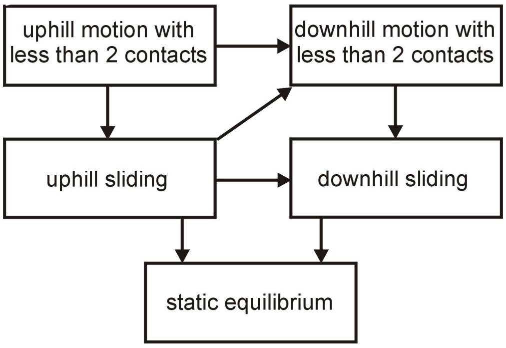

Fig. 2: transition graph of the modes of motion.

IV. DISCUSSION

Assessing the stability of rigid bodies again small dynamic perturbations is a problematic task with very limited results so far. Moreover, all existing results are based on simplified (and quantitatively unreliable) models of frictional impacts, and thus the results themselves are not reliable [4]. Robust stabilization by active control is a natural answer to this problem, which has not received much attention so far. In this paper, we presented a control strategy for the local stabilization of *planar* objects with two contacts. This result is independent of modeling frictional impacts. We hope that our work will inspire the development more general stabilization schemes. The paper is closed by a short discussion of two possible generalizations.

Theorem 1 is restricted to infinitesimal motion because the translation invariance of frictionless dynamics over horizontal lines cannot be used in the case of general surfaces. Thus, the assumptions of small velocities and displacements are crucial in the latter case. Nevertheless, the dynamics of objects supported by *flat slopes* is translation invariant. Thus, our strategy can be used to design globally, spontaneously self-righting objects on flat slopes. To have this property, the object should

(i) have frictionless surface plus two wheels with brakes at the desired contact points

(ii) have a *unique* stable static equilibrium on a horizontal surface with the same contact points to ensure that sliding motion occurs with the desired orientation

(iii) be in statically stable frictional equilibrium on the slope with the same contact points, provided that the brakes are active.

Unfortunately, the transient dynamics preceding sliding motion on two legs may last arbitrarily long in the case of macroscopic motion, even if the perturbation is finite (i.e. only a weaker version of *Proposition 2* applies). Thus, the object may exhibit high transient velocities. In this case, the motion becomes extremely sensitive to unevenness of the slope and neglected factors such as air drag may also have a significant effect on the actual motion. Our strategy also has another practical shortcoming: while the assumption of constant friction coefficient at the legs is acceptable for microscopic motion, μ_i may vary on macroscopic scale along the slope. If μ_i is non-Lipschitz, then the controller may fail to preserve contacts (cf. Appendix V.B.). This problem can be solved by the application of anti-block devices on the wheels.

Stabilization of three-dimensional objects with three (or more) contact points is maybe the practically most important question. *Proposition 1* and *2* can be generalized to three dimensions as long as the motion is infinitesimal; in this case, Q^* is three dimensional; the obstacles form an inverted pyramid and the dynamics ends at the tip of the pyramid within infinitesimal time (which corresponds to sliding on 3 legs in physical space.) However appropriate control strategies to stop the sliding object have not been worked out yet.

V. APPENDIX

A. *Completing the proof of Proposition 1*

Three missing steps of the proof are described here.

Definition of the discrete events

Each impact at obstacle 2 in Q^* is counted as an event, if it follows bouncing from obstacle 1 or sliding along obstacle 1. An example ($t=t_i$) is shown in Fig. 1C.

Proof of property (i)

It is demonstrated below that the impact at $t=t_i$ (event i) reduces the mechanical energy E^* at least by a constant factor c_2, if $h_2 \le h_1$ (see Fig. 1C). In the converse case ($h_2 > h_1$), similar reasoning (not described) guarantees that the last impact to obstacle 1 immediately preceding event i reduces the mechanical energy by a factor c_2. These two results imply property (i) since the mechanical energy of the object is non-increasing at all times.

It is assumed now that $h_2 \le h_1$. Let $\mathbf{v}(t_i^-) = [\dot{q}_1(t_i^-),\ \dot{q}_2(t_i^-)]$ (with dot standing for derivation with respect to time) denote the velocity of the object in configuration space immediately before event i. Then, $\dot{q}_1(t_i^-)$ is positive (since the object arrives from obstacle 1) and $\dot{q}_2(t_i^-)$ is negative (a consequence of $h_2 \le h_1$). Thus the angle between $\mathbf{v}$ and the obstacle exceeds min(α_2,$\pi/2$-α_2) and the normal component of the velocity is

$$|\mathbf{v}_n| \ge \min(\cos\alpha_2, \sin\alpha_2)|\mathbf{v}| \tag{2}$$

In a frictionless collision, $|\mathbf{v}_n|$ is reduced by a factor $-e$ (coefficient of restitution) while the tangential component remains unchanged. Thus, the energy absorption is

$$\Delta E^* = \frac{1}{2}|\mathbf{v}_n|^2(1-e^2) \ge \ge \frac{1}{2}|\mathbf{v}|^2(1-e^2)\min(\cos\alpha_2, \sin\alpha_2)^2 \tag{3}$$

whereas the pre-impact mechanical energy is

$$E^*(t_i^-) = \frac{1}{2}|\mathbf{v}|^2 + h_2\ . \tag{4}$$

Free-falling objects over horizontal terrain fly to a maximum horizontal distance $|\mathbf{v}|^2$ where $\mathbf{v}$ is their velocity at the endpoints of the trajectory (with equality for trajectories having 45° slopes at their endpoints). In our case, the application of this well-known result to the trajectory of the object immediately before event i implies

$|\mathbf{v}|^2 \geq d = h_1\cot\alpha_1 + h_2\cot\alpha_2 \geq h_2(\cot\alpha_1+\cot\alpha_2)$. Thus, (4) takes the form

$$E^*(t_i^-) \leq |\mathbf{v}|^2\left(\frac{1}{2}+\frac{1}{\cot\alpha_1+\cot\alpha_2}\right) \tag{5}$$

Finally, (3) and (5) imply

$$\frac{\Delta E^*}{E^*(t_i^-)} \geq \frac{\frac{1}{2}(1-e^2)\min(\cos\alpha_2, \sin\alpha_2)^2}{\frac{1}{2}+\frac{1}{\cot\alpha_1+\cot\alpha_2}} \tag{6}$$

yielding property (i) .

Proof of property (ii)

We consider the dynamics of the object immediately after event i. It contacts obstacle 2 once or several times before it touches obstacle 1 again. The time of the last contact with obstacle 2 (without hitting obstacle 1) is $t_{i+0.5}$ (Fig. 1C). Between t_i and $t_{i+0.5}$, the object touches only the right side of the wedge. Let an increasing function $p(t)$ denote the total momentum exchange during the time interval (t_i, t) ($t \leq t_{i+0.5}$) between the imaginary point mass and obstacle 2. Since the dynamics is frictionless, the momentum is normal to the contact surface. Thus the configuration of the point mass at time $t_{i+0.5}$ is

$$q_1(t_{i+0.5}) = \dot{q}_1(t_i^+)(t_{i+0.5}-t_i) - \sin\alpha_2 \int_{t_i}^{t_{i+0.5}} p(x)dx \tag{7}$$

$$q_2(t_{i+0.5}) = q_2(t_i) + \dot{q}_2(t_i^+)(t_{i+0.5}-t_i) - \\ -\frac{1}{2}(t_{i+0.5}-t_i)^2 + \cos\alpha_2 \int_{t_i}^{t_{i+0.5}} p(x)dx \tag{8}$$

At the same time we also have $q_1(t_{i+0.5}) = q_2(t_{i+0.5})\cot\alpha_2$ (contact with obstacle 2) and $q_2(t_{i+0.5}) \geq 0$ (object inside wedge) yielding the following inequality after eliminating the integrals from (7) and (8):

$$0 \leq q_2(t_{i+0.5}) = \sin^2\alpha_2 \cdot \Big(-\frac{1}{2}(t_{i+0.5}-t_i)^2 + \\ + \left(\dot{q}_2(t_i^+) + \cot\alpha_2\dot{q}_1(t_i^+)\right)(t_{i+0.5}-t_i) + q_2(t_i)\Big) \tag{9}$$

The velocities $\dot{q}_1$, $\dot{q}_2$ and the height q_2 are bounded by the total mechanical energy $E^*(t_i^+)$ Specifically $\dot{q}_i^2(t_i^+)/2 \leq E^*(t_i^+)$ and $q_2(t_i) \leq E^*(t_i^+)$, leading to

$$0 \leq \sin^2\alpha_2 \cdot \Big(-\frac{1}{2}(t_{i+0.5}-t_i)^2 + \\ + \sqrt{2}(1+\cot\alpha_2)\sqrt{E^*(t_i^+)}(t_{i+0.5}-t_i) + E^*(t_i^+)\Big) \tag{10}$$

The right side of the above equation is quadratic in ($t_{i+0.5}-t_i$) with negative leading coefficient. The positiveness of the right side implies $t_{i+0.5}-t_i \leq c_6(E^*(t_i^-))^{1/2}$ where c_6 is a positive constant. Similarly, the difference $t_{i+1}-t_{i+0.5}$ is bounded from above by $c_7(E^*(t_i^+))^{1/2}$ (detailed calculation omitted). Thus, $t_{i+1}-t_i \leq (c_6+c_7)(E^*(t_i^+))^{1/2}$.

B. *Completing the proof of Theorem 1: the preservation of sliding mode*

It is demonstrated here that the contacts of an object sliding in a given direction are preserved by the controller.

The assumptions of small velocities and displacements imply that the only time-dependent quantities affecting the contact forces $F_i(t)$ and the acceleration of the object during sliding in a given direction are the effective friction coefficients $\nu_i(t)$. The evolution of $\nu_i(t)$ is dictated by the controller based on the feedback variables F_i. An inappropriate controller may eliminate a contact in two ways:

- 'Turnover' occurs if a contact force (F_i) decreases to zero. This state is reached at certain critical values of ν_j, while ν_i does not play any role in turnover, since the brakes at leg i become completely ineffective in the critical state. The ratio F_i/F_j goes to zero at the critical event. If the above ratio decreases below d during the sliding motion, then $d\nu_j/dt$ becomes zero by (1) and only ν_i is varied. As we have seen, the critical state is insensitive to ν_i, i.e. variations of ν_i may not bring the object to the critical state.
- A 'tangential impact' occurs if the contact forces (F_j) diverge to infinity and the sliding motion becomes inconsistent [5]. The contact forces are the solution of a simple system of two linear equations $[\mathbf{a}_1\ \mathbf{a}_2]\cdot\mathbf{f}=\mathbf{g}$ where bold variables denote 2-vectors, specifically $\mathbf{f}=[F_1\ F_2]^T$. By Cramer's rule, $F_1=\det[\mathbf{g}\ \mathbf{a}_2]/\det[\mathbf{a}_1\ \mathbf{a}_2]$, i.e. tangential impact occurs if $\det[\mathbf{a}_1\ \mathbf{a}_2]=0$. The elements of matrix $[\mathbf{a}_1\ \mathbf{a}_2]$ are Lipschitz-continuous functions (formula not shown) of the effective friction coefficients, i.e. also Lipschitz-continuous functions of time. Hence, as the time t^* of the critical event approaches, $F_i(t) > c_8(t^*-t)^{-1}$ with some constant c_8, i.e. the singularity of F_i is nonintegrable corresponding to unbounded energy-absorption. Thus, *sliding is always stopped by the diverging frictional forces before the critical state is reached.*

Hence, the finite, positive contact forces are preserved by the controller.

C. *Completing the proof of Theorem 1: the controller stops the downhill sliding object within $O(c^{-1})$ time*

Let us consider the contact forces $F_i^{(0)}$ of the unperturbed object in equilibrium. Let the tangential to normal force ratio at the legs be $\bullet_i^{(0)}$. By definition, $\bullet_i^{(0)} < \bullet_i$. The brakes are active enough to keep the object in equilibrium (and also to stop sliding) if $\nu_i > \bullet_i^{(0)}$ for both i=1 and 2. As we show, the controller drives the brakes into this region.

As already stated in Appendix V.B, the contact forces of a downhill sliding object depend only on the effective friction coefficients. Let $\nu_i=\mu_i^{(1)}$ and $F_i=F_i^{(1)}$ denote the initial state of the brakes and the corresponding contact forces, when downhill sliding starts. For both pairs of effective friction coefficient: $\bullet_1^{(1)}$, $\bullet_2^{(1)}$ and $\bullet_1^{(0)}$, $\bullet_2^{(0)}$, the corresponding contact forces of are strictly positive. Then, for any constant $0<z<1$, the contact forces $zF_i^{(0)}+(1-z)F_i^{(1)}$ can also be realized by appropriately chosen effective friction coefficients $\mu_i^{(z)}$. This corresponds to a continuous path in the plane of effective friction coefficients connecting $\mu_i^{(0)}$ to $\mu_i^{(1)}$, along which the contact forces are strictly positive. Zero contact forces (i.e. $F_i/F_j=0$) occur along straight lines parallel to the axes, and the critical lines may not cross the above mentioned path. Thus the ratios F_i/F_j and F_j/F_i are strictly positive (and continuous) functions over the rectangular domain $\mu_i^{(0)} < \nu_i < \mu_i^{(1)}$, Both ratios have positive minima over the domains by the Extreme value theorem. If parameter d of the controller is smaller than both minima, then (1) gives a positive lower bound on $d\nu_i(t)/dt \cdot c^{-1}$, i.e the controller reaches the necessarily level of activity within $O(c^{-1})$ time and it also stops the object within a time of the same order.

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