Neutrally floating objects of density 1/2 in three dimensions

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Abstract
This paper is concerned with the Floating Body Problem of S. Ulam: the existence of objects other than the sphere, which can float in a liquid in any orientation. Despite recent results of F. Wegner pointing towards an affirmative answer, a full proof of their existence is still unavailable. For objects with cylindrical symmetry and density ½, the conditions of neutral floating are formulated as an initial value problem, for which a unique solution is predicted in certain cases by a suitable generalization of the Picard-Lindelöf theorem. Numerical integration of the initial value problem provides a rich variety of neutrally floating shapes.

1. Introduction

The equilibria of floating objects subject to gravity and buoyancy forces have intriguing properties. The stable equilibria of symmetrical objects are often asymmetrical [1-5]. Alternatively, the set of equilibrium configurations may have symmetries exceeding the degree of the object's symmetry. An interesting question about floating objects – often referred to as Floating Body Problem – was proposed over seventy years ago by Stanislav Ulam as Problem 19 of the Scottish Book [6]: „are spheres are the only bodies that can float (without turning) in any orientation?" The present paper investigates this question for objects under the influence of gravity and hydrostatic pressure. A simpler two-dimensional version of this problem, also credited to Ulam, concerns the existence of non-circular logs with horizontal axis, which can float in every orientation. There are simple nontrivial solutions among disconnected bodies in two dimensions as well as shapes containing holes in 3 dimensions [7]. To exclude such solutions, both questions are commonly restricted to star-shaped bodies. In this paper, we require solutions to be *simple* with respect to the 'density parameter' $\rho=1/2$ according to

Definition 1: a body is simple with respect to ρ if every planar cut dividing its volume in ratio $\rho:1-\rho$ forms a simply connected set.

Being simple and being star-shaped are closely related and both classes include convex objects. Star-shaped solutions of the planar problem were found long ago for density $\rho=½$ relative to the liquid [8], and much more recently for other densities [9-10], see also [11] for some closely related problems. In both cases, many nontrivial neutrally floating objects have been identified. In three dimensions, there are no solutions in the limit $\rho\to 0$ or 1 [12]; and no solutions among star-shaped objects with central symmetry (other than the sphere) for density $\rho=½$ [13,14]. Nevertheless, F. Wegner has proposed a perturbation expansion scheme starting from the sphere for objects with central symmetry and $\rho\neq½$ [7], as well as for bodies with arbitrary shape and $\rho=½$ [15]. His results point towards the existence of many nontrivial solutions in these wider classes of shapes, even though the proofs are incomplete in that the convergence of the perturbation series has not been examined. Furthermore, no attempt to construct actual solutions of the problem has been reported.

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We take a different approach to construct three-dimensional, neutrally floating objects of density ρ=1/2 with cylindrical symmetry. Our method is an adaptation of [8] to the three-dimensional problem. After reviewing the geometric conditions of neutral floating in Section 2.1-2.2, these are transformed into a non-standard integro-differential equation with given initial conditions (ie. an initial value problem) for the generating curve of the object (Section 2.3-2.4) using fractional order derivatives. It is shown in Section 3 that sufficiently small perturbations of the sphere yield physically meaningful nontrivial solutions of the problem and several examples are constructed by integrating the equations numerically. The paper is closed by a short discussion of related problems.

2. Equations of neutral floating bodies

2.1. Geometric criteria of neutral floating

By the principle of Archimedes, a body of density ρ floats in a liquid of density 1 in such way that a fraction ρ of the object's volume is immersed in the liquid. A configuration satisfying Archimedes' principle is an equilibrium iff the centroid of the object (G) is exactly above the centroid of the immersed portion. The equilibrium is neutral, if after small rotations (with the preservation of Archimedes' principle), the centroid of the immersed part remains on a sphere centered at G, yielding constant potential energy. Our goal is to design objects, for which every configuration satisfying Archimedes' principle is a neutral equilibrium, i.e. for which the centroids of the immersed parts for every possible configuration form a sphere of arbitrary radius r centered at G.

Any plane that divides the object's volume in ratio ρ:1-ρ is called a *water plane* (W) and the intersection of the object with any water plane W a *water section* or W^*. We consider two water planes infinitesimally close to each other. The transformation mapping one (W_1) to the other (W_2) is a rotation by an infinitesimal angle α_{12} about a line l_{12}. The water planes and sections have two remarkable properties described below. For a more detailed description, the reader is advised to consult [1,10] or references therein.

P1: The conservation of the immersed volume implies that l_{12} goes through the centroid of W_1^*. As a consequence, *every water plane is tangential to a closed 'water envelope' surface E formed by the centroids of water sections.* Indeed, E is a wavefront containing singularities rather than a smooth surface, but it has a well-defined tangent everywhere.

P2: If W_1 corresponds to a neutral equilibrium, then the distance between the centroids of the immersed volumes (G_1 and G_2) corresponding to the two water planes is $|G_1G_2| = r\alpha_{12}$. The same distance can also be expressed as $|G_1G_2| = \alpha_{12} I_{12} (\rho V)^{-1}$ where V is the volume of the object and I_{12} is the moment of inertia of W_1^* about the axis l_{12}. Thus neutrally floating bodies are characterized by the additional property that, *the moment of inertia of any water section, about any axis going through its centroid is constant I.*

Property P2 is necessary but not sufficient characterization of a neutral equilibrium since the sphere formed by the centroids is not necessarily centered at G. However, for objects of density ½, G is exactly halfway between the centroid of the submerged part (G_1) and centroid of the rest of the object (G_1') Furthermore, G_1 and G_1' are opposite points of the above mentioned sphere. Hence, the sphere is centered at G, i.e. P1-P2 are necessary and sufficient.

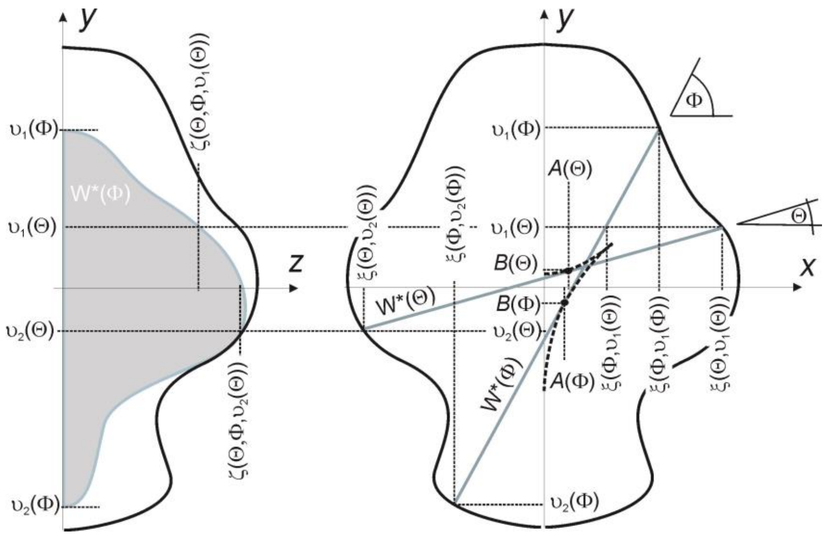

Figure 1: Projections of an object to the *x-y* (right panel) and *y-z* (left panel) plane. Thick solid lines denote its contour, *y* is the symmetry axis. The dashed curve in the right panel is the contour of the water envelope. $W^*(\Theta)$ and $W^*(\Phi)$ are two water sections, which are parallel to the *z* axis. $W^*(\Phi)$ is also shown in the left panel by grey shading. For further notations, see the main text.

2.2. Integral equations of neutral floating

After introducing the notations of the paper, we develop equations corresponding to P1 and P2. We restrict our attention to objects, which are invariant to arbitrary rotation about axis *y* of a Cartesian coordinate system *x-y-z* (Fig. 1). Due to their rotational symmetry, it is enough to consider water planes and sections parallel to the *z* axis. Let $W^*(\Phi)$ denote one such water section, which is at angle $0\le\Phi\le\pi/2$ to the *x-z* plane According to property P1 of Section 2.1, the centroid of $W^*(\Phi)$ belongs to the contour of the rotation-symmetric water envelope *E*. The *x* and *y* coordinates of this point are $A(\Phi)$ and $B(\Phi)$. The intersection of $W^*(\Phi)$ with the *x-y* plane is a line section (right panel of Fig. 1). Let $\upsilon_1(\Phi)$ and $\upsilon_2(\Phi)$ denote the *y* coordinates of its endpoints (υ_2 is usually negative). The functions *A*, *B*, υ_1 and υ_2 determine the object's shape uniquely.

Let $0\le\Theta\le\Phi$. The intersection of $W^*(\Phi)$ with the plane $y=\upsilon_j(\Theta)$ is a line section parallel to the *z* axis. The *x* coordinate of all points along this section is

$$\xi(\Phi,\upsilon_j(\Theta)) = A(\Phi) + \left(\upsilon_j(\Theta) - B(\Phi)\right)\cot\Phi\,. \tag{1}$$

and the half-length of the section is by Pythagoras' theorem:

$$\zeta(\Theta,\Phi,\upsilon_j(\Theta)) = \sqrt{\xi^2(\Theta,\upsilon_j(\Theta)) - \xi^2(\Phi,\upsilon_j(\Theta))} \tag{2}$$

We introduce new variables $\alpha=\sin\Theta$, $\phi=\sin\Phi$ and functions $a(\alpha)=A(\Theta)$; $b(\alpha)=B(\Theta)$; $Y_j(\alpha)=\upsilon_j(\Theta)$; $X(\phi,Y_j(\alpha))=\xi_j(\Phi,\upsilon_j(\Theta))$, which will lead to more convenient equations later. Then, (1), (2) become

$$X(\phi,Y_j(\alpha)) = a(\phi) + \left(Y_j(\alpha) - b(\phi)\right)\phi^{-1}\left(1-\phi^2\right)^{1/2} = \left(\tilde{a}(\phi) + \left(Y_j(\alpha) - b(\phi)\right)\phi^{-1}\right)\left(1-\phi^2\right)^{1/2} \tag{3}$$

$$Z(\alpha,\phi,Y_j(\alpha)) = \sqrt{X^2(\alpha,Y_j(\alpha)) - X^2(\phi,Y_j(\alpha))} \tag{4}$$

where

$$\tilde{a}(\alpha) \overset{def}{=} a(\alpha)\left(1-\alpha^2\right)^{-1/2}. \tag{5}$$

Now we are ready to transform criteria P1 and P2 into equations. By the definition of geometric centroids, P1 takes the form

$$\sum_{j=1}^{2}(-1)^j \int_0^{\phi} Z(\alpha,\phi,Y_j(\alpha))\left(Y_j(\alpha) - b(\phi)\right)Y_j{}'(\alpha)\,d\alpha = 0 \tag{6}$$

where prime (') means derivative. Property P2 applied to an axis parallel to *z* can be expressed as

$$\sum_{j=1}^{2}(-1)^{j+1}\int_0^{\phi} Z(\alpha,\phi,Y_j(\alpha))\left(Y_j(\alpha)-b(\phi)\right)^2 Y_j'(\alpha)d\alpha = I\phi^3 \tag{7}$$

Notice that the left side of the equation is the moment of inertia of a projection of the water section $W^*(\arcsin\phi)$ to the *y*-*z* plane rather than of $W^*(\arcsin\phi)$ itself. This is compensated by the ϕ^3 term on the right side.

Due to the rotational invariance of the object, equilibria are always neutral against an infinitesimal rotation about an axis normal to *z*. Thus, property P2 need not be checked for such axes. In summary, if the water envelope is given, then (6),(7) are necessary and sufficient conditions of neutral floating.

2.3. Steps towards an initial value problem

Analogously to the solution of the planar problem by [8] we first choose a water envelope (see Section 3.3 for more details). Once the functions *a* and *b* have been established, the integral equations (6),(7) depend on values of the functions $Y_j(\alpha)$ over the interval $\alpha\in(0,\phi)$. This observation suggests a transformation of the equations into an initial value problem. (6) and (7) can be written in the general form

$$\sum_{j=1}^{2}(-1)^{j+1}\int_0^{\phi} g_i(\alpha,\phi,Y_j(\alpha))Y_j'(\alpha)d\alpha = f_i(\phi) \tag{8}$$

where f_i are scalar functions and g_i are scalar functionals; i =1 for the first equation and 2 for the second. Differentiating (8) with respect to ϕ yields

$$\sum_{j=1}^{2}(-1)^{j+1}\left(\int_0^{\phi}\frac{\partial g_i(\alpha,\phi,Y_j(\alpha))}{\partial\phi}Y_j'(\alpha)d\alpha + g_i(\phi,\phi,Y_j(\phi))Y_j'(\phi)\right) = f_i'(\phi)\,. \tag{9}$$

If the two by two matrix composed of the elements $g_i(\phi,\phi,Y_j(\phi))$ is nonsingular, then $Y_j'(\phi)$ can be expressed explicitly from the new equations, yielding a first-order initial value problem for $Y_j(\phi)$. Nevertheless it might happen that all elements of the matrix are zero. In this case, the second derivative of (8) becomes

$$\sum_{j=1}^{2}(-1)^{j+1}\left(\int_0^{\phi}\frac{\partial^2 g_i(\alpha,\phi,Y_j(\alpha))}{\partial\phi^2}Y_j'(\alpha)d\alpha + \left.\frac{\partial g_i(\alpha,\phi,Y_j(\alpha))}{\partial\phi}\right|_{\alpha=\phi} Y_j'(\phi)\right) = f_i''(\phi) \tag{10}$$

which is again a candidate for an initial value problem. If the $\partial g_i()/\partial\phi$ terms also happen to be zero, additional derivation of the equations might be necessary. Unfortunately, this method fails for the specific function g_i of the problem of neutral floating, because $g_i(\phi,\phi,Y_j(\phi))$ is identically zero whereas the first derivative $\partial g_i/\partial\phi$ does not exists; specifically

$$\lim_{\alpha\to\phi}\left|\frac{\partial g_i(\alpha,\phi,Y_j(\alpha))}{\partial\phi}\right| = \infty \tag{11}$$

The diverging limit indicates that taking the second derivative of (8) is “too much”, whereas the first derivative is not enough. This special property of g_i is a consequence of the square-root type singularity of the function *Z* in (4) at $\alpha=\phi$, inherited by g_i. The specific form of *Z* simplies that *the fractional derivative of order 3/2* of g_i is finite and nonzero at $\alpha=\phi$; thus the 3/2th derivative of (8) leads to an initial value problem.

2.4. Calculation of the fractional derivative

Fractional derivatives are defined as integer order derivatives of a fractional integral of order less than 1 [16]. Thus, the first step towards the 3/2th derivative is to take the semi-integral of (8) using the definition of Riemann–Liouville differintegrals:

$$\sum_{j=1}^{2}(-1)^{j+1}\int_0^{\chi}(\chi-\phi)^{-1/2}\cdot\int_0^{\phi}g_i(\alpha,\phi,Y_j(\alpha))Y_j'(\alpha)d\alpha d\phi=\int_0^{\chi}(\chi-\phi)^{-1/2}\cdot f_i(\phi)d\phi \tag{12}$$

Before proceeding with the main steps, the order of integration is changed on the left side of the equation and the functions G_i and F_i are introduced:

$$\sum_{j=1}^{2}(-1)^{j+1}\int_0^{\chi}Y_j'(\alpha)\cdot\underbrace{\int_{\alpha}^{\chi}(\chi-\phi)^{-1/2}g_i(\alpha,\phi,Y_j(\alpha))d\phi}_{G_i(\alpha,\chi,Y_j(\alpha))}d\alpha=\underbrace{\int_0^{\chi}(\chi-\phi)^{-1/2}\cdot f_i(\phi)d\phi}_{F_i(\chi)} \tag{13}$$

We differentiate both sides with respect to χ, using the Leibniz integral rule:

$$\sum_{j=1}^{2}(-1)^{j+1}\left(\int_0^{\chi}Y_j'(\alpha)\cdot\frac{\partial}{\partial\chi}G_i(\alpha,\chi,Y_j(\alpha))d\alpha+Y_j'(\chi)\cdot\underbrace{G_i(\chi,\chi,Y_j(\chi))}_{\text{zero}}\right)=F_i'(\chi) \tag{14}$$

The term $G_i(\chi,\chi,Y_j(\chi))$ equals zero (see (43) in Appendix A.1). Thus we have,

$$\sum_{j=1}^{2}(-1)^{j+1}\int_0^{\chi}Y_j'(\alpha)\cdot\frac{\partial}{\partial\chi}G_i(\alpha,\chi,Y_j(\alpha))d\alpha=F_i'(\chi) \tag{15}$$

Differentiating both sides once more yields

$$\sum_{j=1}^{2}(-1)^{j+1}\left(\int_0^{\chi}Y_j'(\alpha)\cdot\frac{\partial^2}{\partial\chi^2}G_i(\alpha,\chi,Y_j(\alpha))d\alpha+Y_j'(\chi)\cdot\frac{\partial}{\partial\chi}G_i(\alpha,\chi,Y_j(\alpha))\bigg|_{\alpha=\chi}\right)=F_i''(\chi)\,. \tag{16}$$

The functions $F_i''(\chi)$ can be expressed in closed form, specifically $F_1''(\chi)=0$ and $F_2''(\chi)=8I\chi^{3/2}$. Thus, the unknowns $Y_j'(\chi)$ can be expressed explicitly from (16) as

$$\mathbf{Y}'(\chi)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\mathbf{A}(\chi,\mathbf{Y}(\chi))^{-1}\left(\begin{bmatrix}0\\8I\chi^{3/2}\end{bmatrix}-\int_0^{\chi}\mathbf{C}(\alpha,\chi,\mathbf{Y}(\alpha),\mathbf{Y}'(\alpha))d\alpha\cdot\begin{bmatrix}1\\-1\end{bmatrix}\right) \tag{17}$$

where **Y** and **Y**' are column vectors composed of the functions Y_j and Y_j'; **A** and **C** are 2 by 2 matrices with elements

$$a_{ij}(\chi,Y_j(\chi))=\frac{\partial}{\partial\chi}G_i(\alpha,\chi,Y_j(\alpha))\bigg|_{\alpha=\chi} \tag{18}$$

$$c_{ij}(\alpha,\chi,Y_j(\chi),Y_j'(\chi))=Y_j'(\alpha)\cdot\frac{\partial^2}{\partial\chi^2}G_i(\alpha,\chi,Y_j(\alpha)) \tag{19}$$

A and **C** will be examined thoroughly to study the solutions of the initial value problem (17).

3. Solutions

3.1. The existence and uniqueness of solutions

Spheres of any radius R are neutrally floating objects. They correspond to $Y_j(\phi)=R(-1)^{j+1}\phi$. We deduce implicitly that this function satisfies the initial value problem (17) for $a(\phi)=b(\phi)\equiv0$ with initial conditions $Y_j(0)=b(0)$. We refer to the corresponding equations and solutions as well as elements of these equations as *trivial* equations, solutions, etc. In this section, we want to examine nontrivial solutions obtained by minor perturbations of $a(\phi)$ and $b(\phi)$.

There are some technical issues arising at χ=0. Every solution (including the family of trivial solutions) satisfies

$$Y_j(0)=0 \tag{20}$$

i.e. is (20) not an appropriate initial condition. Indeed, if $\chi \to 0^+$, all elements **A**, **C** and **F** in (6), (7) go to zero, which is inherited by (17). Hence, (17) becomes identity. This means that the initial condition must include $Y_j'(0)$. For example, the trivial solutions have $\mathbf{Y}'(0)=R\cdot[1\ -1]^T$. To avoid difficulties at $\chi=0$, we narrow our focus to those solutions, which coincide with the trivial solution ($a(\phi)=b(\phi)=0$ and $Y_j(\phi)=R(-1)^{j+1}\phi$) if ϕ is below some positive value α_1. It is demonstrated below that the problem has a unique solution under this restriction.

It should also be noticed that (17) has been obtained by taking the 3/2[th] derivative of an equation of the form $u(\chi,\mathbf{Y}(\chi),...)=0$. The transformed equation admits false solutions for which $u(\chi,\mathbf{Y}(\chi),...)=constant\cdot\chi^{3/2}$ rather than 0. Nevertheless, the set of false solutions correspond to water sections with moment of inertia $I+constant\cdot\chi^{-3/2}$, which contradicts any initial condition of the form $Y_j'(0)=constant$. Hence all solutions of our initial value problem satisfy (6)-(7).

Lemma 1 states that any solution of the perturbed equations is necessarily close to the trivial solution, without examining if such solutions exist or not. The questions of existence and uniqueness are answered by *Lemma* 2. The two lemmas are summarized in *Theorem 1*, the main result of the paper.

Lemma 1: for any given scalar $0<\alpha_1$, there exist positive scalars k and ε_0, such that if
(i) $a(\alpha)=0$, $b(\alpha)=0$ and $Y_j(\alpha)=(-1)^{j+1}\alpha$ if $0\le\alpha\le\alpha_1$;
(ii) the absolute values of $\tilde{a}(\alpha)$, $b(\alpha)$, and of their derivatives up to third order exist and are $<\varepsilon<\varepsilon_0$ for any $\alpha_1\le\alpha\le 1$
then any solution of equations (17) *over the interval $\alpha_1<\alpha\le 1$ satisfies*

$$\left|Y_j'(\alpha)-(-1)^{j+1}\right| \le k\varepsilon e^{k\alpha} \tag{21}$$

Proof of Lemma 1:
We arrive to (21) via proof by contradiction. The initial section $0\le\alpha<\alpha_1$ of $Y_j(\alpha)$ satisfies (21) for any k by point (i) of *Lemma 1*. Let us assume now that (21) is violated no matter how large k is. Then there must exist a unique scalar $\alpha_1<\chi(k)<1$ for any k such that (21) holds for $\alpha\le\chi(k)$ and there is equality in (21) for $\alpha=\chi(k)$ and j=1 or 2. In this case we also have

$$\left|Y_j(\alpha)-(-1)^{j+1}\alpha\right| \le \int_0^\alpha \left|Y_j'(\beta)-(-1)^{j+1}\right| d\beta < \varepsilon\int_0^\alpha ke^{k\beta}d\beta = \varepsilon\left(e^{k\alpha}-1\right) < \varepsilon e^{k\alpha} \quad if \quad \alpha\le\chi(k)\,. \tag{22}$$

From this point, the argument k of χ is dropped for brevity.

If ε is small enough, then (21) and (22) imply that

1) each entry of $\mathbf{A}(\chi,\mathbf{Y}(\chi))$ is within a neighborhood of radius $*\cdot\varepsilon e^{k\chi}$ of its trivial value, and the trivial value is bounded (Appendix A.1); * represents some finite positive scalar, which is independent of k. Furthermore, $\mathbf{A}(\chi)$ is non-singular, i.e. $|\det\mathbf{A}|$ has a positive lower bound (Appendix A.2). The two results imply that $\mathbf{A}^{-1}(\chi,\mathbf{Y}(\chi))$ is also within a neighborhood of radius $*\cdot\varepsilon e^{k\chi}$ of its bounded trivial value.
2) if k>1, then the second derivative of G_i is within distance $*\cdot\varepsilon e^{k\alpha}$ of its bounded trivial value:

$$\frac{\partial^2}{\partial\chi^2}G_i(\alpha,\chi,Y_j(\alpha)) \in DDG_{ij}^{(0)}(\alpha,\chi) \pm *\cdot\varepsilon e^{k\alpha}, \tag{23}$$

see Appendix A.3. By plugging (21) and (23) into (19) one obtains

$$c_{ij} \in \left((-1)^{i+1} \pm k\varepsilon e^{k\alpha}\right)\left(DDG_{ij}^{(0)}(\alpha,\chi) \pm *\varepsilon e^{k\alpha}\right) \in .... \tag{24}$$

$$\underbrace{(-1)^{i+1}DDG_{ij}^{(0)}(\alpha,\chi,Y_i)}_{\text{bounded trivial value}} \pm \underbrace{\left(\max\left|DDG_{ij}^{(0)}(\alpha,\chi)\right| + *\cdot\underbrace{k^{-1}}_{<1} + *\cdot\underbrace{\varepsilon e^{k\alpha}}_{<<1}\right)\cdot\varepsilon ke^{k\alpha}}_{*\varepsilon k\exp(k\alpha)\ deviation}$$

Hence, we conclude that each entry of $\mathbf{C}(\alpha,\chi,\mathbf{Y}(),\mathbf{Y}'())$ is within a neighborhood of radius $*\cdot\varepsilon k e^{k\alpha}$ of its bounded trivial value.

Eq. (17) together with the bounds of $\mathbf{A}^{-1}$ and $\mathbf{C}$ found above, imply that (21) holds if $\alpha=\chi$ with the left hand side strictly smaller than the right-hand side, provided that k exceeds some threshold that we denote by k_0. This result contradicts the assumption that we have equality in (21) if $\alpha=\chi$. Hence, (21) is true for all χ if $k>k_0$. Details of the last piece of calculation are omitted, but we point out that $\mathbf{C}$ is inside an integral in (17). Integrating its $*\varepsilon k\exp(k\alpha)$ maximum deviation from the trivial value yields $*\varepsilon\exp(k\chi)$ maximum deviation in $\mathbf{Y}'$•

Lemma 2: there exists a positive scalar ε_0 such that (i) and (ii) of Lemma 1 imply that (17) has a unique solution.

Proof of Lemma 2: ODE's with Lipschitz-continuous right-hand sides and given initial condition have unique solutions according to the Picard-Lindelöf theorem [17]. We sketch an adaptation of the standard proof of this result to the initial value problem (17).

By introducing the function $\mathbf{\Psi}()=\mathbf{Y}'()$, (17)-(19) can be rewritten as

$$\mathbf{\Psi}(\chi)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\mathbf{A}\left(\chi,\int_0^\chi\mathbf{\Psi}(\beta)d\beta\right)^{-1}\left(\begin{bmatrix}0\\8I\chi^{3/2}\end{bmatrix}-\int_0^\chi\mathbf{C}\left(\alpha,\chi,\int_0^\alpha\mathbf{\Psi}(\beta)d\beta,\mathbf{\Psi}(\alpha)\right)\cdot d\alpha\cdot\begin{bmatrix}1\\-1\end{bmatrix}\right) \tag{25}$$

$$a_{ij}=\frac{\partial}{\partial\chi}G_i\left(\alpha,\chi,\int_0^\alpha\Psi_j(\beta)d\beta\right)\Bigg|_{\alpha=\chi} \tag{26}$$

$$c_{ij}=\Psi_j(\alpha)\cdot\frac{\partial^2}{\partial\chi^2}G_i(\alpha,\chi,\int_0^\alpha\Psi_j(\beta)d\beta) \tag{27}$$

Assume that the solution $\mathbf{\Psi}$ of (25) is known for $\chi\le\chi_0$ and satisfies (21):

$$\left|\Psi_j(\chi)-(-1)^{j+1}\cos\chi\right|\le k\varepsilon e^{k\chi} \tag{28}$$

We wish to prove that there is a unique solution over an additional finite interval $\chi_0<\chi\le\chi_1$. Splitting the integrals in (25) at χ_0 yields

$$\mathbf{\Psi}(\chi)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\mathbf{A}\left(\chi,\underbrace{\int_0^{\chi_0}\mathbf{\Psi}(\beta)d\beta}_{known}+\int_{\chi_0}^{\chi}\mathbf{\Psi}(\beta)d\beta\right)^{-1}\cdot\ldots \tag{29}$$

$$\ldots\left(\underbrace{\begin{bmatrix}0\\8I\chi^{3/2}\end{bmatrix}-\int_0^{\chi_0}\mathbf{C}\left(\alpha,\chi,\int_0^\alpha\mathbf{\Psi}(\beta)d\beta,\mathbf{\Psi}(\alpha)\right)\cdot\begin{bmatrix}1\\-1\end{bmatrix}d\alpha}_{known}-\int_{\chi_0}^{\chi}\mathbf{C}\left(\alpha,\chi,\int_0^\alpha\mathbf{\Psi}(\beta)d\beta,\mathbf{\Psi}(\alpha)\right)d\alpha\cdot\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

The right-hand side of (29) defines a self-map K of the space S_0 of vector valued continuous functions $\mathbf{\Psi}(\chi)$ over the interval $\chi\in(\chi_0,\chi_1)$. S_0 and the metric d induced by the norm

$$\left\|\mathbf{\Psi}(\chi)\right\|\overset{def}{=}\max_{\chi,j}\left|\Psi_j(\chi)\right| \tag{30}$$

form a Banach space. Let S denote the closed subset of S_0 determined by (28). The arguments used in the proof of *Lemma 1* imply that for ε small enough and $k>k_0$, K maps S into itself. The contraction principle implies that if K is a contraction then it has a unique fixed point, corresponding to a unique solution of (29). Repeated application of the above argument yields global existence and uniqueness for $\alpha_1\le\chi\le1$. Integrating the solution $\mathbf{\Psi}$ leads to a unique solution $\mathbf{Y}$ of the original problem.

The only remaining gap in the proof is the contractivity of K. $\mathbf{A}$ is Lipschitz in its second variable (cf. (46), details omitted). As $\mathbf{A}$ is nonsingular (Appendix A.2), its inverse is also Lipschitz with some

Lipschitz constant L_{invA}. Similarly, (27) and some examination of (49) yield that **C** is Lipschitz-continuous functional of $\mathbf{\Psi}(\chi)$, $\chi\in(\chi_0,\chi_1)$ with a Lipschitz constant L_C (details omitted).

Next, we consider two elements $\mathbf{\Psi}^{(1)}$ and $\mathbf{\Psi}^{(2)}$ of the set S. Then,

$$d\left(\mathbf{A}\left(\chi,\int_0^{\chi_0}\mathbf{\Psi}(\beta)d\beta+\int_{\chi_0}^{\chi}\mathbf{\Psi}^{(1)}(\beta)d\beta\right)^{-1},\mathbf{A}\left(\chi,\int_0^{\chi_0}\mathbf{\Psi}(\beta)d\beta+\int_{\chi_0}^{\chi}\mathbf{\Psi}^{(2)}(\beta)d\beta\right)^{-1}\right)\le L_{invA}(\chi-\chi_0)d\left(\mathbf{\Psi}^{(1)},\mathbf{\Psi}^{(2)}\right) \tag{31}$$

and

$$d\left(\int_{\chi_0}^{\chi}\mathbf{C}\left(\alpha,\chi,\int_0^{\chi_0}\mathbf{\Psi}(\beta)d\beta+\int_{\chi_0}^{\chi}\mathbf{\Psi}^{(1)}(\beta)d\beta,\mathbf{\Psi}^{(1)}(\alpha)\right)d\alpha,\int_{\chi_0}^{\chi}\mathbf{C}\left(\alpha,\chi,\int_0^{\chi_0}\mathbf{\Psi}(\beta)d\beta+\int_{\chi_0}^{\chi}\mathbf{\Psi}^{(2)}(\beta)d\beta,\mathbf{\Psi}^{(1)}(\alpha)\right)d\alpha\right)\le\ldots \tag{32}$$

$$\ldots L_C(\chi-\chi_0)d\left(\mathbf{\Psi}^{(1)},\mathbf{\Psi}^{(2)}\right)$$

These inequalities and the boundedness of all terms in the formula of K imply that $d(K(\mathbf{\Psi}^{(1)}),K(\mathbf{\Psi}^{(2)}))\le L(\chi-\chi_0)d(\mathbf{\Psi}^{(1)},\mathbf{\Psi}^{(2)})$ with some constant L (details omitted) . Hence, if $\chi-\chi_0<L^{-1}$ then K is contractive. •

The two lemmas yield

Theorem 1: for any given scalars $0<\alpha_1<1$ and $\Delta>0$, there exists a positive scalar ε_0 such that (i) and (ii) of Lemma 1 imply that (17) has a unique solution $\mathbf{Y}(\chi)$*;* $Y_j(\chi)=(-1)^{j+1}\chi$ for $0\le\chi\le\alpha_1$ and $|Y_j(\chi)-(-1)^{j+1}\chi|<\Delta$ *for* $\alpha_1<\chi\le1$. *This solution satisfies* (6), (7).

3.2. Characterization of acceptable solutions

The water envelopes $a(\phi)$ and $b(\phi)$ together with a pair of function $Y_j(\phi)$ over the interval $0\le\phi\le1$ determine the 'upper' (j=1) and the 'lower' (j=2) half of a unique curve in the x-y plane. The coordinates of a point $P_j(\phi)$ of the curve are $x=(-1)^{j+1}X(\phi,Y_j(\phi))$; $y=Y_j(\phi)$; $z=0$. Rotation of this curve about the y axis generates a unique object with cylindrical symmetry. Below we state a sufficient condition under which the object is simple. This condition is satisfied by the nontrivial solutions predicted by *Theorem 1*.

Lemma 3: If

$$\left.\begin{array}{ll}\textit{either} & (-1)^{j+1}Y_j(\phi)>\delta\\ \textit{or} & (-1)^{j+1}X(\phi,Y_j(\phi))>\delta\end{array}\right\}\ \textit{for any}\ 0<\phi<1 \tag{33}$$

with

$$\delta=\max\left\{\begin{array}{l}\max_\phi|a(\phi)|\\ \max_\phi|b(\phi)|\end{array}\right\} \tag{34}$$

then the object is a simple topological ball.

Proof of Lemma 3:

The condition of the lemma means that one can draw a square of size $2\delta\times2\delta$ about the origin of the x-y plane such that the water envelope is inside the square while the contour curve is outside (Fig. 2). Outside the square, the upper-right quarter of the x-y plane is covered with the non-intersecting lines $L(\phi)$ each containing a point $P_1(\phi)$ of the generating curve. The lower-right quarter contains the points $P_2(\phi)$ each one lying on the mirror images of lines $L(\phi)$ about the y axis. Thus, all points $P_j(\phi)$ for $\phi<1$ are in the right half-plane, separated from the y axis. Furthermore, two points of the contour curve corresponding to different values of ϕ or different values of j, lie on different lines, hence they may not coincide. Altogether we have found that the *contour curve does not touch the y axis* (except at the endpoints: ϕ=1), and *it is not self-intersecting* (or self-touching). Rotating such curves generates topological balls.

Due to the cylindrical symmetry of the object, being simple is equivalent of requiring that $L^*(\phi)$ is a connected line segment for every ϕ (rather than the union of multiple segments). $L^*(0)$ is connected, hence, the object is simple iff by varying ϕ, the topology of $L^*(\phi)$ does not change. A topological change of L^* occurs at ϕ if the contour curve touches $L(\phi)$ at $P_1(\phi)$ or the mirror image of $L(\phi)$ at $P_2(\phi)$ *without crossing the line*. Nevertheless this situation is impossible because, as already mentioned, the points $P_j(\phi)$ lie on the lines $L(\phi)$ and their mirror images, which are free of intersections outside the square.•

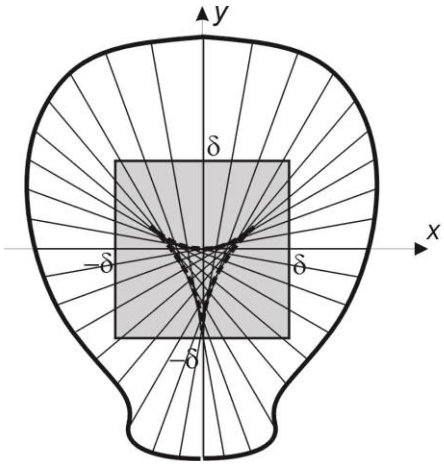

Fig. 2: illustration of *Lemma 3*: if the water envelope is inside the grey square, and the contour curve is outside, then the object is a simple topological ball.

3.3. Water envelopes and numerical examples

A convenient way to find a suitable water envelope is to pick a C^1 function $\rho(\Phi)$ with a bounded but possibly discontinuous second derivative representing the signed radius of curvature of the water envelope at tangent angle Φ. Then,

$$A(\Phi) = \int_{\Phi}^{\pi/2} \rho(\Theta)\cos\Theta d\Theta + c_A \tag{35}$$

$$B(\Phi) = \int_{\Phi}^{\pi/2} \rho(\Theta)\sin\Theta d\Theta + c_B \tag{36}$$

The symmetry of the problem dictates that

A: $A(\pi/2)=0$, hence $c_A=0$;
B: variations of c_B result in translated copies of the same envelope, i.e. we can choose $c_B=0$.
C: $A(0)=0$, which is a constraint for $\rho(\Phi)$ by **(35)**;
D: $\rho(\Phi)$ is π-periodic and even;
E: $\rho(\Phi-\pi/2)$ is odd, implying $\rho(\pi/2)=0$.

In the variables α and ϕ, **(35)** and **(36)** become

$$a(\phi) = \int_{\phi}^{1} \rho(\arcsin\alpha) d\alpha \tag{37}$$

$$b(\phi) = \int_{\phi}^{1} \rho(\arcsin\alpha) \frac{\alpha}{\sqrt{1-\alpha^2}} d\alpha \tag{38}$$

The $\arcsin\alpha$ function has a square-root singularity at $\alpha=1$. According to observation E, $\rho(\arcsin\alpha)\approx constant\cdot(\pi/2-\alpha)^{1/2}$ near $\phi=1$. This singularity is cancelled by a $(\pi/2-\alpha)^{-1/2}$ term in **(38)**, thus $b(\phi)$ becomes C^2 with a bounded third derivative. At the same time, $a(\phi)\approx constant\cdot(1-\phi^2)^{3/2}$ near $\phi=1$, which means that $a(\phi)$ is singular at $\phi=1$, but the related function $\tilde{a}(\phi)$ (see (5)) has a bounded third derivative. The smoothness of b and $\tilde{a}$ mean that any function $\rho(\Phi)$ multiplied by a sufficiently small constant meets condition (ii) of *Lemma 1*.

Two examples fulfilling the above requirements are

$$\rho(\Phi) = c \cdot \cos((2n+1)\Phi) \qquad n = 1,2,3\ldots \tag{39}$$

$$\rho(\Phi) = c \cdot \begin{cases} 0 & if \quad \phi \le \pi/4 \\ \sin^2 4\Phi - \frac{85}{84}\sin^3(4\Phi) & if \quad \pi/4 < \phi \le \pi/2 \end{cases} \tag{40}$$

where c is an arbitrary constant; the number 85/84 is determined by Observation C. The second example obeys condition (i) of the lemma, hence this envelope generates a nontrivial solution by *Theorem 1* if its unspecified constant is small enough, see also Fig. 3A .

The first example does not meet condition (i) nevertheless the solution appears to exist and to be unique in this case, too (Fig. 3B-D). Indeed, condition (i) is probably unnecessary for *Lemma 1*, but it simplifies the proof (see Appendix A.3). Additionally, condition (i) has a central role in the proof of *Lemma 2*. Nevertheless, existence (and uniqueness) of the solution might be provable with a different approach without condition (i).

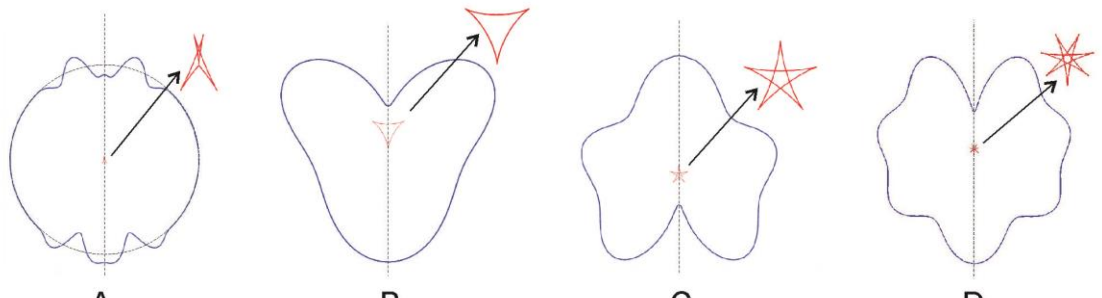

Figure 3: Numerically determined contour curves and water envelopes of some neutrally floating shapes with cylindrical symmetry. A: water envelope (34) with *c*=0.5; B-D: water envelope (33) with *n*=1,2,3 and *c*=0.5; 0.5;0.4. In all cases, $Y_1'(0)=-Y_2'(0)=1$.

4. Discussion

This paper is concerned with the proof of existence and the construction of neutrally floating, simple objects of density 1/2 (other than the sphere) in three dimensions. As we show, there are many solutions even among bodies with cylindrical symmetry. Our study leaves many open questions, including the necessity of condition (i) in *Theorem 1*, or the existence of solutions for densities other than 1/2.

The present discussion of the Floating Body Problem concentrates on gravitational (and buoyancy) forces, and excludes any other forces acting on the object. A different approach has been taken by R. Finn and coworkers [18,19], see also [20,21], who studied particles floating in gravity-free environment under the effect of capillary forces. In this approach, the contact angle of the object and the liquid is a free parameter analogous to density in the presence of gravity. The two-dimensional capillary floating problem admits nontrivial solutions similarly to the Archimedean version, see [22] for more background. In three dimensions, only a special nonexistence result has been published: spheres are the only objects, which can float in any orientation in such a way that the capillary forces generate *a perfectly flat liquid surface* around the object. In most cases, a macroscopically flat liquid surface typically becomes distorted in a small neighborhood of a floating object to minimize the surface energy of the system. This more general situation seems to be unexplored.

While gravity-free floating may appear as a weird setting at first sight, it is physically as relevant as the Archimedean approach. Physical systems under terrestrial conditions are inevitably subject to both gravity and capillary forces. The relative strengths of the two forces are determined by the dimensionless Eötvös- (or Bond-) number of the system. Small-scale objects have low Eötvös numbers (indicating the dominance of capillary effects), whereas upscaling an object increases the Eötvös number. For example, the Eötvös number of a ball of density ½ and radius r floating in water is approximately $(r/4\text{mm})^2$. Thus, the dominance of each of the two effects can be realized in a physical experiment. Additionally, there exists a generalized – and completely unexplored – version of Ulam's problem, which seeks neutrally floating objects under dual influence of gravity and surface tension for given density, contact angle and Eötvös number.

A Appendix

The appendix contains several technical results needed for *Lemma 1*.

A.1 Deviation of a_{ij} from its trivial value

a_{ij} is given by (18) as a partial derivative of G_i, which is defined by an improper integral (13). We define a new function

$$Q(\alpha,\phi,Y_j(\alpha)) \overset{def}{=} \begin{cases} Z(\alpha,\phi,Y_j(\alpha))^2\left(1-\dfrac{\alpha}{\phi}\right)^{-1} & if \quad \alpha<\phi \\ \lim\limits_{\beta\to\phi^-} Q(\beta,\phi,Y_j(\beta)) & if \quad \alpha=\phi \end{cases} \tag{41}$$

which proves useful later, see Appendix A.3. This definition is motivated by the square-root type singularity of Z at $\alpha=\phi$, which implies that Q is bounded and strictly positive. With the new function and equations (6)-(8),(13) we obtain

$$G_i(\alpha,\chi,Y_j(\alpha)) = \int_\alpha^\chi (\chi-\phi)^{-1/2}\left(1-\frac{\alpha}{\phi}\right)^{1/2}\sqrt{Q(\alpha,\phi,Y_j(\alpha))}\big(Y_j(\alpha)-b(\phi)\big)^i)\,d\phi \tag{42}$$

The variable ϕ of integration is changed to $\Gamma=(\chi-\phi)^{1/2}\cdot(\chi-\alpha)^{-1/2}$:

$$G_i(\alpha,\chi,Y_j(\alpha)) = 2(\chi-\alpha)\int_0^1\left(\chi+\alpha\frac{\Gamma^2}{1-\Gamma^2}\right)^{-1/2} Q^{1/2}\big(\alpha,\chi-\Gamma^2(\chi-\alpha),Y_j(\alpha)\big)\big(Y_j(\alpha)-b\big(\chi-\Gamma^2(\chi-\alpha)\big)\big)^i\,d\Gamma \tag{43}$$

This form of G_i is a proper integral, and also free of terms diverging to infinity at $\alpha=\chi$. a_{ij} can now be calculated from (18) and (43) by using the Leibniz rule, then by plugging $\alpha=\chi$, and finally by evaluating a simple integral:

$$a_{ij}\big(\chi,Y_j(\chi)\big) = \frac{\pi}{2}\chi^{-1/2}Q^{1/2}\big(\chi,\chi,Y_j(\chi)\big)\big(Y_j(\chi)-b(\chi)\big)^i \tag{44}$$

The formula above contains $Q(\chi,\chi,Y_j(\chi))$, which can be expressed as a function of $X()$:

$$\begin{aligned} Q\big(\chi,\chi,Y_j(\chi)\big) &= \lim_{\alpha\to\chi^-}\frac{X^2(\alpha,Y_j(\alpha))-X^2(\chi,Y_j(\alpha))}{1-\alpha/\chi} = \dots \\ &-\chi\lim_{\alpha\to\chi}\frac{X^2(\chi,Y_j(\alpha))-X_j^2(\alpha,Y_j(\alpha))}{\chi-\alpha} = \dots \\ &-\chi\frac{\partial}{\partial\chi}X^2(\chi,Y_j(\alpha))\Big|_{\alpha=\chi} \end{aligned} \tag{45}$$

We plug this equation into (44) and use (3) to express a_{ij} explicitly as a function of Y_j:

$$a_{ij}\left(\chi, Y_j(\chi)\right) = \frac{\pi}{\sqrt{2}}\left(-\chi \left.\frac{\partial X^2(\chi, Y_j(\alpha))}{\partial \chi}\right|_{\alpha=\chi}\right)^{1/2}\left(Y_j(\chi) - b(\chi)\right)^i = \tag{46}$$

$$= \frac{\pi}{2}\sqrt{-\frac{\partial}{\partial \chi}\left[\left(\tilde{a}(\chi)\chi + Y_j(\alpha) - b(\chi)\right)^2\left(\chi^{-2} - 1\right)\right]\Big|_{\alpha=\chi}} \cdot \left(Y_j(\chi) - b(\chi)\right)^i$$

Eq. (22) and point (ii) of *Lemma 1* can be expressed as: $Y_j(\chi) \in (-1)^{j+1}\chi \pm \varepsilon\exp(k\chi)$ and $\tilde{a}(\chi), b(\chi), \tilde{a}'(\chi)$, $b'(\chi) \in \pm\varepsilon \in \pm\varepsilon\exp(k\chi)$. Plugging these into (46) together with the inequality $\chi \geq \alpha_1$; replacing higher order terms in ε by a small constant time ε; and noting that the term under the square-root sign has a strictly positive lower bound lead to the final expression

$$a_{ij}\left(\chi, Y_j(\chi)\right) \in (-1)^{i(j+1)}\frac{\pi}{\sqrt{2}}\chi^{i-1/2} \pm k_1 \varepsilon e^{k\chi} \tag{47}$$

with some positive constant k_1 not specified for brevity. This is the result we had to prove.

A.2 A lower bound of detA

An approximation of a_{ij} with $*\varepsilon\exp(k\chi)$ uncertainty has been given by (47). This formula yields

$$\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \in \pi^2\chi^2 \pm {*}\varepsilon\exp(k\chi) \tag{48}$$

Since $\chi > \alpha_1$, we have found a positive lower bound of det**A**.

A.3 The second derivative of G_i

This section is devoted to the proof of equation (23). The second derivative of G_i is calculated from (43) by successive applications of the Leibniz rule. The result (calculated by Maple software and not shown) can be written in the form

$$\frac{\partial^2 G_i(\alpha, \chi, Y_j(\alpha))}{\partial \chi^2} = \int_0^1 \sum_k \left(W_{1k} \cdot W_{2k} \cdot \ldots\right) d\Gamma \tag{49}$$

where W_{lk} are functions of $i, j, \alpha, \chi, \Gamma, Y_j(\alpha)$, $a()$ and $b()$. Specifically, they include constants and eight non-constant terms listed in Table 1.

Table 1: a full list of terms emerging in the second partial derivative of G_i.

Number	Term
1	$\left(\chi + \alpha\frac{\Gamma^2}{1-\Gamma^2}\right)^{-1/2}$
2	$Y_j(\alpha) - b\left(\chi - \Gamma^2(\chi - \alpha)\right)$
3	$b'\left(\chi - \Gamma^2(\chi - \alpha)\right)$
4	$b''\left(\chi - \Gamma^2(\chi - \alpha)\right)$
5	$Q\left(\alpha, \phi, Y_j(\alpha)\right)^{1/2}$ where $\phi = \chi - \Gamma^2(\chi - \alpha)$
6	$Q\left(\alpha, \phi, Y_j(\alpha)\right)^{-1/2}$ where $\phi = \chi - \Gamma^2(\chi - \alpha)$
7	$\left(1-\Gamma^2\right)\left.\frac{\partial Q\left(\alpha, \phi, Y_j(\alpha)\right)}{\partial \phi}\right\|_{\phi=\chi-\Gamma^2(\chi-\alpha)}$
8	$\left(1-\Gamma^2\right)^2\left.\frac{\partial^2 Q\left(\alpha, \phi, Y_j(\alpha)\right)}{\partial \phi^2}\right\|_{\phi=\chi-\Gamma^2(\chi-\alpha)}$

As we show below, each term has a bounded absolute value, and its deviation from its trivial value is at most *constant*·$\varepsilon\exp(k\alpha)$:

- **Term 1** is not bigger than $\chi^{-1/2}$, hence it is bounded from above by the constant $\alpha_1^{-1/2}$; this term is not affected by perturbations of the water envelope, since it does not depend on any of the functions Y_j a or b.
- **Term 2** is $Y_j(\alpha)-b\left(\chi-\Gamma^2(\chi-\alpha)\right)\in(-1)^{j+1}\alpha\pm\varepsilon e^{k\alpha}\pm\varepsilon\in(-1)^{j+1}\alpha\pm 2\varepsilon e^{k\alpha}$
- **Term 3 and 4** are $\in 0\pm\varepsilon\in 0\pm\varepsilon e^{k\alpha}$
- **Term 5 and 6**: it is enough to show that $Q(...)$ itself has an absolute value bounded from above *and below* by positive bounds, and that it is affected by at most a constant times $\varepsilon\exp(k\alpha)$ by the perturbation. By using (3), (4) and (41), expanding the nominator and ordering its terms into pairs (marked by square brackets), Q can be expressed as

$$Q\left(\alpha,\phi,Y_j(\alpha)\right)=\frac{\left(\tilde a(\alpha)+\left(Y_j(\alpha)-b(\alpha)\right)\alpha^{-1}\right)^2\left(1-\alpha^2\right)-\left(\tilde a(\phi)+\left(Y_j(\alpha)-b(\phi)\right)\phi^{-1}\right)^2\left(1-\phi^2\right)}{1-\alpha/\phi}=\ldots \tag{50}$$

$$=\phi\frac{\left[\tilde a(\alpha)^2-\tilde a(\phi)^2\right]-\left[\alpha^2\tilde a(\alpha)^2-\phi^2\tilde a(\phi)^2\right]+Y_i^2(\alpha)\left[\alpha^{-2}-\phi^{-2}\right]+etc.}{\phi-\alpha}=\ldots$$

$$=\frac{\left[\overbrace{\left(\tilde a(\alpha)-\tilde a(\phi)\right)}^{\in\pm\varepsilon(\phi-\alpha)}\overbrace{\left(\tilde a(\alpha)+\tilde a(\phi)\right)}^{\in\pm2\varepsilon}\right]\phi-\left[\overbrace{\left(\alpha\tilde a(\alpha)-\phi\tilde a(\phi)\right)}^{\in\pm2\varepsilon(\phi-\alpha)}\overbrace{\left(\alpha\tilde a(\alpha)+\phi\tilde a(\phi)\right)}^{\in\pm2\varepsilon}\right]\phi+\left[\overbrace{\phi Y_j^2(\alpha)\left(\alpha^{-2}-\phi^{-2}\right)}^{\in\left(1+\frac{\alpha}{\phi}\right)(\phi-\alpha)\pm\varepsilon e^{k\alpha}2\alpha_1^{-3}(\phi-\alpha)}\right]+etc.\ \ldots}{\phi-\alpha}$$

  Only a few terms of the nominator are shown. Each square bracket in the nominator can be expressed as $*\cdot(\phi-\alpha)$ in order to cancel the denominator. This step is straightforward in some cases, but less so in others. For example, in the case of the first two square brackets, we use that $\left|\tilde a(\alpha)\right|,\left|\tilde a'(\alpha)\right|<\varepsilon$; at the third one, we exploit that

  - $Y_j(\alpha)=(-1)^{j+1}\alpha$ if $\alpha\le\alpha_1$
  - (22) holds and $\alpha^{-2}-\phi^{-2}\in\pm2\alpha_1^{-3}(\phi-\alpha)$ if $\alpha>\alpha_1$.

  This calculation boils down to

$$Q(\alpha,\phi,Y_j(\alpha))\in 1+\alpha/\phi\pm*\cdot\varepsilon\exp(k\alpha). \tag{51}$$

- **Term 7:** The trivial value of Q is given by (51). This is used to determine the trivial value of term 7, which is bounded because

$$\left(1-\Gamma^2\right)\left.\frac{\partial Q\left(\alpha,\phi,Y_j(\alpha)\right)}{\partial\phi}\right|_{\phi=\chi-\Gamma^2(\chi-\alpha)}=\frac{\phi-\alpha}{\chi-\alpha}\cdot\left(-\frac{\alpha}{\phi^2}\right)=\ldots \tag{52}$$

$$=\begin{cases}\dfrac{\phi-\alpha}{\chi-\alpha}\dfrac{\alpha}{\phi}\phi^{-1}\le 1\cdot1\cdot\left(\dfrac{\alpha_1}{2}\right)^{-1} & if\ \ \alpha\ge\chi/2\\[2ex] \dfrac{\phi-\alpha}{\phi}\dfrac{\alpha}{\phi}(\chi-\alpha)^{-1}\le 1\cdot1\cdot\left(\dfrac{\alpha_1}{2}\right)^{-1}\le & if\ \ \alpha\le\chi/2\end{cases}$$

  The deviation of term 7 from its trivial value can be obtained analogously to the calculations of term 5 and 6 by exploiting that the first *and second* derivatives of $\tilde a$ and b are $\in\pm\varepsilon$. These are not shown for brevity.
- **Term 8:** the trivial value can be investigated using a formula analogous to (52):

$$\left(1-\Gamma^2\right)^2 \cdot \frac{\partial Q\left(\alpha,\phi,Y_j(\alpha)\right)}{\partial\phi}\Bigg|_{\phi=\chi-\Gamma^2(\chi-\alpha)} = \left(\frac{\phi-\alpha}{\chi-\alpha}\right)^2 \frac{2\alpha}{\phi^3} = \ldots \tag{53}$$

$$= \begin{cases} 2\left(\frac{\phi-\alpha}{\chi-\alpha}\right)^2 \frac{\alpha}{\phi}\phi^{-2} \le 2\cdot 1\cdot 1\cdot\left(\frac{\alpha_1}{2}\right)^{-2} & if\ \alpha \ge \chi/2 \\ 2\left(\frac{\phi-\alpha}{\phi}\right)^2 \frac{\alpha}{\phi}(\chi-\alpha)^{-2} \le 2\cdot 1\cdot 1\cdot\left(\frac{\alpha_1}{2}\right)^{-2} & if\ \alpha \le \chi/2 \end{cases}$$

As before, explicit bounds of the nontrivial value are not shown. They can be obtained by exploiting that the derivatives of $\tilde{a}$ and b up to third order are $\in\pm\varepsilon$.

Both of these properties are inherited by the second derivative of G_i according to eq. (49), yielding (23).

References

[1] E. N. Gilbert, How things float. *Amer. Math. Monthly* 98: 201–216 (1991).

[2] P. Erdős, G. Schibler, R. C. Herndon, Floating equilibrium of symmetrical objects and the breaking of symmetry. Part 1: Prisms. *American Journal of Physics* 60: 335-345 (1992).

[3] P. Erdős, G. Schibler, R. C. Herndon, Floating equilibrium of symmetrical objects and the breaking of symmetry. Part 2: The cube, the octahedron, and the tetrahedron. *American Journal of Physics* 60: 345-356 (1992).

[4] D. W. Bass, Stability of icebergs. *Annals of Glaciology* 1: 43-47 (1980).

[5] J. F. Nye and J. R. Potter, The use of catastrophe theory to analyze the stability and toppling of icebergs. *Annals of Glaciology* 1: 49-54 (1980).

[6] R. D. Mauldin, *The Scottish Book, Mathematics from the Scottish Café*, Birkhauser, Boston (1981).

[7] F. Wegner, Floating Bodies of Equilibrium in Three Dimensions. The central symmetric case. e-Print archive arXiv:0803.1043 (2008).

[8] H. Auerbach, Sur un probleme de M. Ulam concernant I'équilibre des corps flottants, *Studia Math* 7: 121-142 (1938).

[9] F. Wegner, Floating Bodies of Equilibrium. *Studies in Applied Mathematics* 111: 167-183 (2003).

[10] F. Wegner, Floating Bodies of Equilibrium in 2D, the Tire Track Problem and Electrons in a Parabolic Magnetic Field. e-Print archive arXiv:physics/0701241v3 (2007).

[11] P. L. Várkonyi, Floating body problems in two dimensions. *Studies in Applied Mathematics* 122: 195-218 (2009).

[12] L. Montejano, On a problem of Ulam concerning a characterization of the sphere, *Studies in Appl. Math*. LIII: 243-248 (1974).

[13] R. Schneider, Functional equations connected with rotations and their geometrical applications, *L'enseignement math*. 16: 297-305 (1970).

[14] K. J. Falconer, Applications of a Result on Spherical Integration to the Theory of Convex Sets, *Amer. Math. Monthly*, 90: 690-693 (1983).

[15] F. Wegner, Floating bodies of equilibrium at density 1/2 in arbitrary dimensions. e-Print archive arXiv:0902.3538 (2009).

[16] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley & Sons (1993).

[17] Earl A. Coddington, N. Levinson *Theory of Ordinary Differential Equations*, McGraw-Hill, New York (1955).

[18] R. Finn, Floating Bodies Subject to Capillary Attractions, *J. Math. Fluid Mech.* 11: 443–458 (2009).

[19] R. Finn and M. Sloss, Floating Bodies in Neutral Equilibrium, *J. Math. Fluid Mech.* 11: 459–463 (2009).

[20] E. Gutkin, Capillary floating and the billiard ball problem. *J. Math. Fluid Mech.*14: 363-382. DOI: 10.1007/s00021-011-0071-0 (2012).

[21] Raphael E., di Meglio J-M., Berger M., and Calabi E. Convex particles at interfaces, *J. Phys. I France* 2: 571-579 (1992).
[22] E. Gutkin, Addendum to: Capillary floating and the billiard ball problem, e-Print archive arXiv:1206.2272 (2012).