SIMPLE MODELS FOR DEGENERATE CUSP CATASTROPHES

Zsolt Gaspar[1] and Peter Varkonyi[2]

ABSTRACT

The paper shows simple two-degrees-of-freedom mechanical models to illustrate the unstable-X point of bifurcation, the stable-X point of bifurcation and the point-like instability. Using the total potential energy function we determine: the critical loads of the perfect structures, the transformations which separate the active and passive parts of the energy functions, the different types of the equilibrium paths and the imperfection-sensitivity surfaces for all the three models.

Key Words: *instability, cusp catastrophe, imperfection-sensitivity*

1 INTRODUCTION

The behaviour of an engineering structure depends only on a single control parameter, the time (the variation of the loads and imperfections also occur in time). According to Thom's theorem only *fold catastrophe* can arise typically: i. e. a real structure (with its inevitable manufacturing imperfections) always loses its stability at a fold catastrophe, in which the equilibrium path merely reaches a maximum load at a limit point.

The structure and the load are often assumed to be symmetric. In this case a Taylor expansion of the potential function in some (suitable) variables must contain vanishing coefficients of odd-power terms. Hence in the case of perfect structures, a fold catastrophe cannot arise, but perfect symmetry on the drawing board can give rise to the *cusp catastrophe*.

Let us analyse a finite degree of freedom elastic structure with a one-parameter load. The structure is in equilibrium if the gradient of the total potential energy function is zero, and the load is critical when the determinant of the Hessian is also zero. In the critical state the energy function has a catastrophe point. In the neighbourhood of this catastrophe point the function can be transformed into a canonical form of the catastrophe.

[1] Professor, Budapest University of Technology and Economics, Hungarian Academy of Sciences, Hungary
[2] PhD student, Budapest University of Technology and Economics, Hungary

Changing the load parameter $\left(\Lambda, \text{ or } \lambda = \Lambda - \Lambda^{cr}\right)$ means moving along a curve in the parameter space even if we introduce some imperfection parameters. The structure might lose its stability only if this curve reaches the bifurcation set of the examined catastrophe, which is well-known from the catastrophe theory. A subclassification of the catastrophes can be done by the help of these curves (also called λ-routes in the following).

In the case of the fold catastrophe only two subclasses exist. The load curve either crosses the bifurcation set (which is a point in the one dimensional parameter space) or it turns back when it arrives at the bifurcation set. The first case is the limit point of the equilibrium path; the second case is the asymmetric point of bifurcation. Both cases were analyzed by Koiter [1].

For the cusp catastrophes the dimension of the parameter space is two (the axes of the space will be denoted by *a* and *b*), and the bifurcation set is a cusped curve, which separates the parameter plane into two parts. An energy function belongs to every point of the parameter space, and the number of its stationary points is different for the different regions of the plane.

There are two types of cusp catastrophes: the dual and the standard cusp.

In the case of the dual cusp the points below the bifurcation set belong to functions having one minimum and two maxima, while the other part gives functions with one maximum. The structure can lose its stability only if it had stability. So if $\lambda < 0$ (but it is close to 0), we must be below the curve of the bifurcation set. (There is no minimum, i. e. no stable equilibrium above the curve.)

There are two possibilities for the perfect structure to lose its stability (Figure 1):

- the λ-route crosses the bifurcation set at the cusp point (unstable-symmetric point of bifurcation) or
- it turns back (unstable-X point of bifurcation).

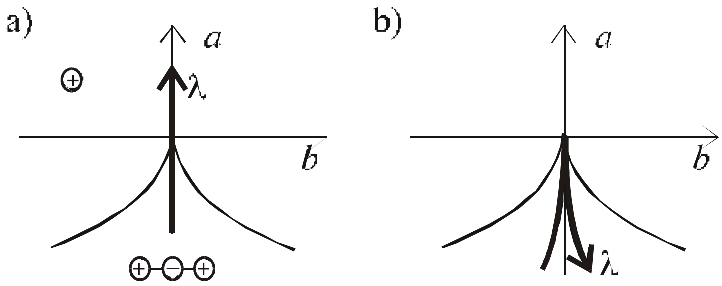


Fig. 1 – Routes in control space through dual cusp catastrophe
(the thin curves are the points of the bifurcation set)

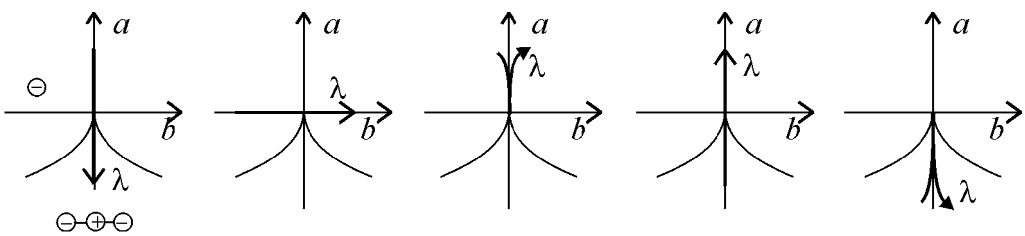


Fig. 2 – Routes in control space through standard cusp catastrophe

In the case of the standard cusp the points below the bifurcation set belong to functions having one maximum and two minima, while the other part gives functions with one minimum. So we can start from any part of the parameter space. Theoretically there are five possibilities for the perfect structure to arrive at the cusp point (Figure 2):

- starting from above the curve the λ-route crosses the bifurcation set (stable-symmetric point of bifurcation),
- the smooth λ-route remains always above the bifurcation set (cut-off point),
- the λ-route turns back (point-like instability), or
- starting from below the curve the λ-route crosses the bifurcation set (upside down case),
- it turns back (stable-X point of bifurcation).

Koiter [1] has analyzed both the unstable- and stable-symmetric point of bifurcation. Thompson and Hunt [2] dealt with the cut-off point, Gaspar [3] with the upside down case, so we will show models for the remaining three degenerate cases.

2. UNSTABLE-X POINT OF BIFURCATION

Let us consider a simplified version of the structure analyzed by Gaspar and Domokos [4]. The structure is a hinged cantilever (Figure 3) comprising a link with normal rigidity *k*=4, pinned to the rigid foundation and supported by a linear rotational spring of stiffness *c*=1. The vertical load acts on the top of the link. A state of the structure can be given by two state variables (φ, h). The unloaded perfect structure is in equilibrium in the state $\varphi = 0,\ h = 1$. We introduce two imperfections: the rotational spring is unstressed if $\varphi = \varepsilon_1$, and normal rigidity of the link is $k + \varepsilon_2$.

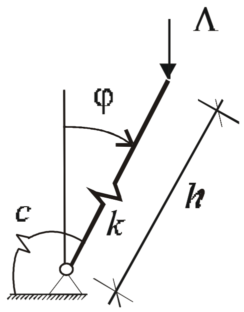


Fig. 3 – Model for the unstable-X point of bifurcation

The total potential energy function is

$$V(\varphi, h, \Lambda, \varepsilon_1, \varepsilon_2) = \frac{1}{2}c(\varphi - \varepsilon_1)^2 + \frac{1}{2}(k + \varepsilon_2)(1-h)^2 + \Lambda h \cos\varphi . \qquad (1)$$

The perfect structure is in equilibrium if

$$V_\varphi = c\varphi - \Lambda h \sin\varphi = 0 \qquad (2)$$

$$V_h = k(h-1) + \Lambda\cos\varphi = 0 , \qquad (3)$$

which are fulfilled when $\varphi = 0$ and $h = 1 - \Lambda / k$. The Hessian matrix on the primary equilibrium path is diagonal:

$$\mathbf{H}_0 = \langle c - \Lambda(1 - \Lambda / k) \quad k \rangle , \qquad (4)$$

so it is singular if

$$\Lambda_{1,2}^{cr} = \frac{k}{2} , \qquad (5)$$

i. e. the critical state is given by the following data:

$$\Lambda^{cr} = 2, \quad \varphi^{cr} = 0, \quad h^{cr} = 0.5 . \tag{6}$$

We slip the origin into the critical point by the linear transformations

$$\Lambda = \Lambda^{cr} + \lambda, \quad \varphi = \varphi, \quad h = h^{cr} + v , \tag{7}$$

and determine the truncated Taylor series:

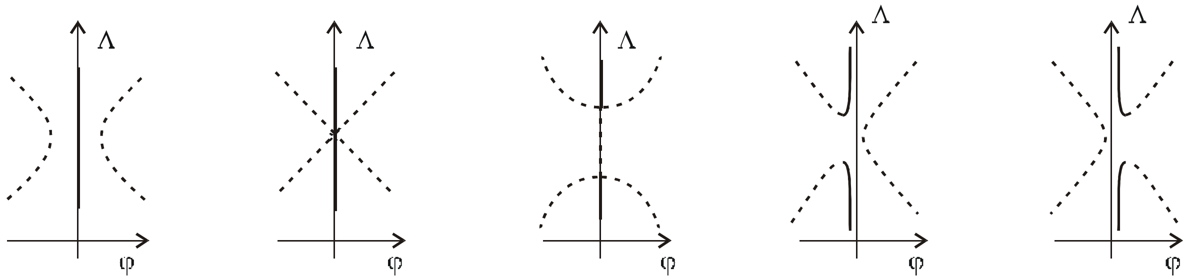
$$V(\varphi, v, \lambda, \varepsilon_1, \varepsilon_2) = \frac{1}{24}\varphi^4 - \left(1 + \frac{\lambda}{2}\right)\varphi^2 v + \frac{1}{4}\lambda\varphi^2 + 2v^2 + \left(\lambda - \frac{\varepsilon_2}{2}\right)v - \varepsilon_1\varphi . \tag{8}$$

Both φ and v appears in the second term, but the diffeomorphism

$$\varphi = \varphi, \quad v = \frac{u\sqrt{2}}{2} + \left(\frac{1}{4} + \frac{\lambda}{8}\right)\varphi^2 - \frac{\lambda}{4} + \frac{\varepsilon_2}{8} \tag{9}$$

splits V into an active and a passive part:

$$V(\varphi, u, \lambda, \varepsilon_1, \varepsilon_2) = -\frac{1}{12}\varphi^4 + \frac{\lambda^2 - \varepsilon_2}{8}\varphi^2 - \varepsilon_1\varphi + u^2 . \tag{10}$$



$\varepsilon_1=0,\ \varepsilon_2<0$ | $\varepsilon_1=0,\ \varepsilon_2=0$ | $\varepsilon_1=0,\ \varepsilon_2>0$ | $\varepsilon_1<0,\ \varepsilon_2=0$ | $\varepsilon_1>0,\ \varepsilon_2=0$

Fig. 4 – Equilibrium paths for different imperfections (unstable-X point)

Using the active part the equilibrium paths can be given as

$$\lambda = \pm\left(\frac{4\varphi^3 + 3\varepsilon_2\varphi + 12\varepsilon_1}{3\varphi}\right)^{1/2} \tag{11}$$

and they are illustrated in Figure 4 in the case of different imperfections.

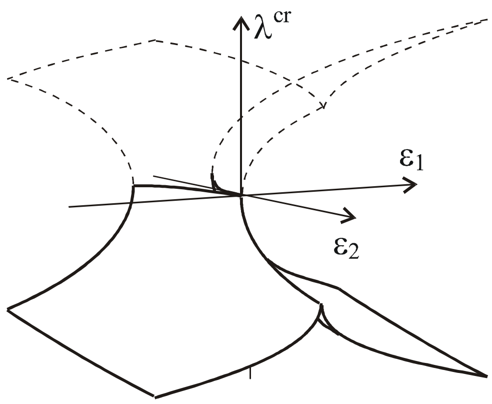


Fig. 5 – The imperfection-sensitivity surface (unstable-X point)

Eliminating φ from equations $V_\varphi = 0, \quad V_{\varphi\varphi} = 0$ we get the imperfection-sensitivity surface (Figure 5):

$$\lambda^{cr} = -\left(\varepsilon_2 + (12\varepsilon_1)^{2/3}\right)^{1/2} , \tag{12}$$

which shows that there are

- non-dangerous imperfections when the structure does not lose its stability,
- very dangerous imperfections, e. g. the exponent is ½ if $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, and what is more the exponent is 1/3 if $\varepsilon_2 = 0$.

3. POINT-LIKE INSTABILITY

The structure shown in Figure 6 consists of two telescopic members (without friction), two linear and two rotational springs. The unloaded perfect structure is free of stress, the linear springs are in vertical, and the telescopic members are in horizontal position. The structure is loaded by a vertical dead load of magnitude Λ. We want to determine the equilibrium paths and the imperfection-sensitivity surface in a small vicinity of the critical point of the perfect structure when

$$c_1 = 1,\ c_2 = 3,\quad l = 1 + \varepsilon_1$$

and we introduce also a horizontal load of magnitude ε_2.

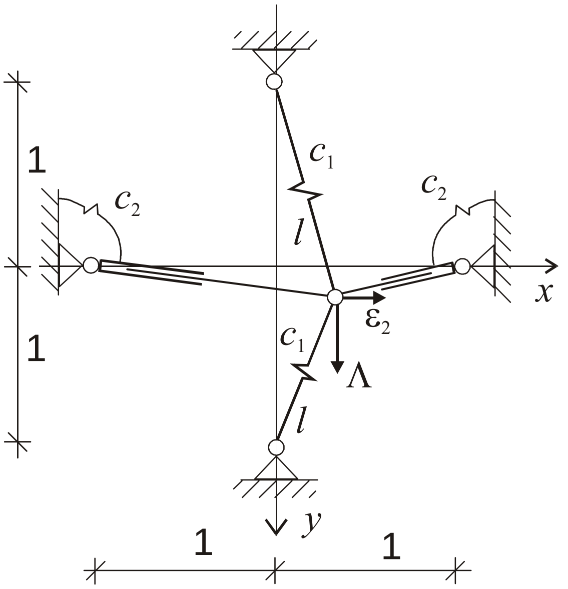


Fig. 6 – The model for the point-like instability

The position of the structure can be uniquely described by the *x* and *y* coordinates of the middle hinge. The total potential energy function of the imperfect structure is

$$V(x, y, \Lambda, \varepsilon_1, \varepsilon_2) = \frac{1}{2}\left[c_1\left(\sqrt{x^2 + (1+y)^2} - (1+\varepsilon_1)\right)^2 + c_1\left(\sqrt{x^2 + (1-y)^2} - (1+\varepsilon_1)\right)^2 + \right.$$

$$\left. + c_2 \arctan^2 \frac{y}{1+x} + c_2 \arctan^2 \frac{y}{1-x} \right] - \Lambda y - \varepsilon_2 x . \tag{13}$$

The position $x = y = 0$ is critical for the unloaded perfect structure hence the Hessian is singular:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}. \tag{14}$$

At this point $\Lambda^{cr}=0$, so now $\Lambda=\lambda$.

The truncated Taylor series of the potential function of the imperfect structure is

$$V=\left(\frac{1}{4}+\frac{\varepsilon_1}{4}\right)x^4+(8-\varepsilon_1)x^2y^2-2y^4-\varepsilon_1x^2+4y^2-\lambda y-\varepsilon_2x. \tag{15}$$

This has a mixed term, so we use the diffeomorphism

$$x=x,\quad y=v+\frac{1}{8}\lambda\left(-1+\frac{1}{8}\varepsilon_1\right)x^2+\left(-1+\frac{1}{8}\varepsilon_1\right)x^2v$$

to split the passive and active parts of the energy function:

$$V^p(v)=-2v^4+4v^2-\lambda v, \tag{17}$$

$$V^a(x,\lambda,\varepsilon_1,\varepsilon_2)=\frac{1}{4}x^4+\left(\frac{1}{8}\lambda^2-\varepsilon_1\right)x^2-\varepsilon_2x. \tag{18}$$

The passive part can be transformed into a Morse saddle. The active part can be induced from the canonical form

$$f(x,a,b)=\frac{1}{4}x^4+\frac{1}{2}ax^2+bx \tag{19}$$

of the cusp catastrophe by the transformations

$$a=\frac{1}{4}\lambda^2-2\varepsilon_1,\quad b=-\varepsilon_2, \tag{20}$$

so the λ-route of the perfect structure looks like the third case in Figure 2.

If $\varepsilon_2=0$ and $x=0$, then $V_x^a=0$ so the primary equilibrium path is vertical (Figure 7a-c). All the points (but the critical one) of the path are stable for the perfect structure (Figure 7a), this is why this case is called point-like instability. If $\varepsilon_1<0$ then the critical point disappears (Figure 7b), and there will be an interval of the load where three equilibrium positions exist if $\varepsilon_1>0$ (Figure 7c).

If $\varepsilon_2\neq 0$ then the paths can be given by the following formula:

$$\lambda=\pm 2\sqrt{\frac{\varepsilon_2+2\varepsilon_1x-x^3}{x}}. \tag{21}$$

So there is always a continuous stable equilibrium path (Figure 7d), but if ε_1 is large enough then a separate closed path also appears (Figure 7e).

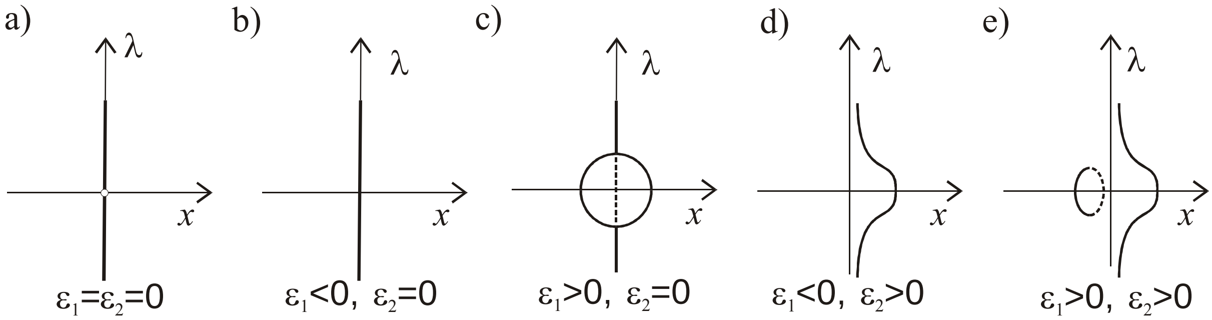


Fig. 7 – Equilibrium paths at different imperfections (point-like instability)

Substituting the transformations (20) into the equations of the bifurcation set:

$$a=-3p^2,\quad b=2p^3 \tag{22}$$

we get the equation of the imperfection-sensitivity surface in parametric form:

$$\varepsilon_1=\frac{1}{8}\left(\lambda^{cr}\right)^2+3p^2,\qquad \varepsilon_2=-2p^3. \tag{23}$$

The first bifurcation point decreases when $\varepsilon_2=0$ and $\varepsilon_1>0$. In this case

$$\lambda^{cr}=-\sqrt{8\varepsilon_1}, \tag{24}$$

but in other cases the structure will not lose its stability for continuous change of load, so the imperfection-sensitivity surface shows unimportant points (Figure 8).

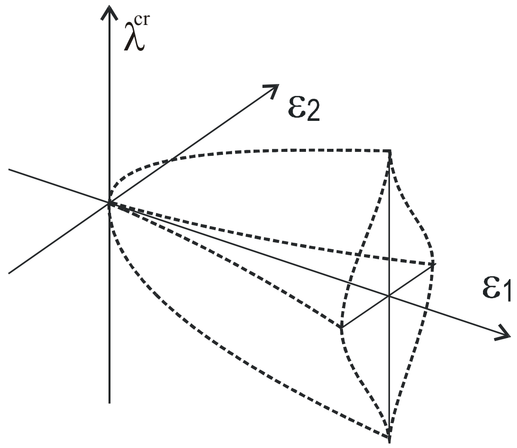


Fig. 8 – Imperfection-sensitivity surface (point-like instability)

4. STABLE-X POINT OF BIFURCATION

The structure shown in Figure 9 consists of two linear springs. c_i denotes the stiffness in both tension and compression, and l_i is the stress free length of the ith spring (i=1, 2). The structure is loaded by a vertical dead load of magnitude Λ. We want to determine the equilibrium paths and the imperfection sensitivity surface in the small vicinity of the critical point of the perfect structure in the case of the following values:

$$c_1 = 1.985177808 + \varepsilon_1,\ c_2 = 1,\ l_1 = 0.4,\ l_2 = 0.7\,. \tag{25}$$

A horizontal load of magnitude ε_2 is also applied as an imperfection.

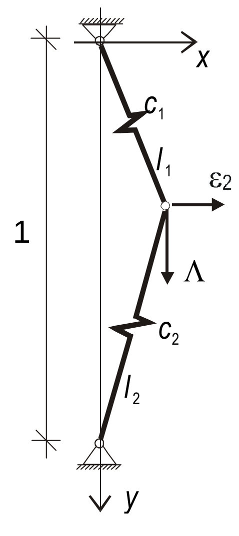


Fig. 9 – Model for the stable-X point of bifurcation

The position of the structure can be uniquely described by the x and y coordinates of the middle hinge. The unloaded perfect structure has three equilibrium positions:

$$x_1 = 0,\ y_1 = 0.3665011578,\ x_{2,3} = \pm 0.2185749299,\ y_{2,3} = 0.335. \tag{26}$$

The first position is unstable, the other two are stable. We have to know in which stable state the structure is when we start to load it. Let us suppose, that the structure is in the state of positive x.

Increasing the load the perfect structure arrives at a critical state when

$$\Lambda^{cr} = 0.4455533422,\ x^{cr} = 0,\ y^{cr} = 0.5157563683. \tag{27}$$

We slip the origin into the critical point $\left(\Lambda = \Lambda^{cr} + \lambda,\ x = x,\ y = y^{cr} + u\right)$ and determine the truncated Taylor series:

$$V(x,u,\lambda,\varepsilon_1,\varepsilon_2) = 1.494072598x^4 - 5.976290393x^2u^2 + 1.492588904u^2 + 0.1122200088\varepsilon_1 x^2 + \\ + (0.1157563683\varepsilon_1 - \lambda)u - \varepsilon_2 x. \tag{28}$$

Using the diffeomorphism

$$x = x,\ u = 0.3349884209\lambda - 0.03877704303\varepsilon_1 + w + (1.341285652\lambda - 0.1552623560\varepsilon_1)x^2 + \\ + 2.001988082x^2 w \tag{29}$$

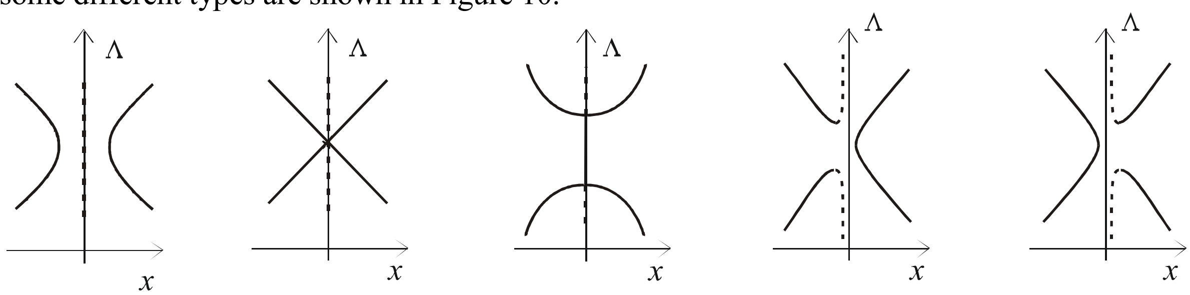
we can split the function into an active and a passive part:

$$V(x,w,\lambda,\varepsilon_1,\varepsilon_2) = 1.494072598x^4 + (0.1122200088\varepsilon_1 - 0.6706428261\lambda^2)x^2 + \\ + 1.492588904w^2 - \varepsilon_2 x. \tag{30}$$

The equilibrium paths can be given as

$$\lambda = \pm\left(\frac{4.455643272x^3 + 0.1673320051\varepsilon_1 x - 0.7455533417\varepsilon_2}{x}\right)^{1/2}, \tag{31}$$

some different types are shown in Figure 10.



$\varepsilon_1<0,\ \varepsilon_2=0$ $\varepsilon_1=0,\ \varepsilon_2=0$ $\varepsilon_1>0,\ \varepsilon_2=0$ $\varepsilon_1=0,\ \varepsilon_2>0$ $\varepsilon_1=0,\ \varepsilon_2<0$

Fig. 10 – Equilibrium paths at different imperfections (stable-X point)

The imperfection-sensitivity surface (Figure 11) is given by the following function:

$$\lambda^{cr} = \pm 0.4090623487\left(\varepsilon_1 + 15.28075669\varepsilon_2^{2/3}\right)^{1/2}. \tag{32}$$

The surface is similar to that shown in Figure 5, but stable and unstable positions are changed and we started in a position with positive x, so only the left (belongs to negative ε_2-s) part of the imperfection-sensitivity surface is interesting.

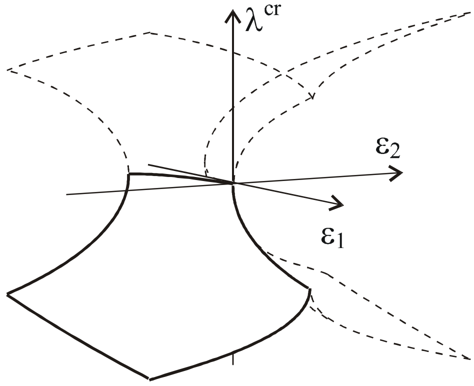


Fig. 11 – The imperfection sensitivity surface (stable-X bifurcation)

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