Different Affine Decomposition of the Model of the Prototypical Aeroelastic Wing Section by TP model transformation

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***Abstract* — The Tensor Product (TP) model transformation is a recently proposed technique for transforming given Linear Parameter Varying (LPV) models into affine model form, namely, to parameter varying convex combination of Linear Time Invariant (LTI) models. The main advantage of the TP model transformation is that the Linear Matrix Inequality (LMI) based control design frameworks can immediately be applied to the resulting affine models to yield controllers with tractable and guaranteed performance. The effectiveness of the LMI design depends on the LTI models of the convex combination. Therefore, the main objective of this paper is to study how the TP model transformation is capable of determining different types of convex hulls of the LTI models. The study is conducted trough the example of the prototypical aeroelastic wing section.**

I. INTRODUCTION

The affine model form is a dynamic model representation whereupon LMI based control design techniques can immediately be executed. It describes given LPV models by a parameter varying convex combination of LTI models. The TP model form is a kind of affine decomposition, where the convex combination is defined by one variable weighting functions of each parameter separately. Convex optimization or linear matrix inequality based control design techniques can immediately be applied to affine, hence to TP models [5, 8, 12]. An important advantage of the TP model representation is that the convex hull defined by the LTI models can readily be modified and analyzed via the one variable weighting functions. Furthermore, the feasibility of the LMI's can be considerably relaxed by modifying the type of the resulting convex hull.

The TP model transformation is a recently proposed numerical method to transform LPV models into TP model form [3, 4]. It is capable of transforming different LPV model representations (such as physical model given by analytic equations, fuzzy, neural network, genetic algorithm based models) into TP model form in a uniform way. In this sense it replaces the analytical derivations and affine decompositions (that could be a very complex or even an unsolvable task). Execution of the TP model transformation takes a few minutes by a regular Personal Computer. The TP model transformation minimizes the number of the LTI components of the resulting TP model. Furthermore, the TP model transformation is capable of resulting different types of convex hulls of the given LPV model.

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In this paper we study how the TP model transformation is applicable to generate different types of convex hulls of the given LPV models. The study is conducted through the example of the prototypical aeroelastic wing section.

II. PRELIMINARIES

A. Linear Parameter-Varying state-space model

Consider the following parameter-varying state-space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t), \qquad (1)$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t),$$

with input $\mathbf{u}(t)$, output $\mathbf{y}(t)$ and state vector $\mathbf{x}(t)$. The system matrix

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) \\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t)) \end{pmatrix} \in \mathbb{R}^{O\times I} \qquad (2)$$

is a parameter-varying object, where $\mathbf{p}(t) \in \Omega$ is time varying N-dimensional parameter vector, and is an element of the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$. $\mathbf{p}(t)$ can also include some elements of $\mathbf{x}(t)$.

B. Convex state-space TP model

$\mathbf{S}(\mathbf{p}(t))$ can be approximated for any parameter $\mathbf{p}(t)$ as the convex combination of LTI system matrices $\mathbf{S}_r$, $r = 1, \ldots, R$. Matrices $\mathbf{S}_r$ are also called *vertex systems*. Therefore, one can define weighting functions $w_r(\mathbf{p}(t)) \in [0,1] \subset \mathbb{R}$ such that matrix $\mathbf{S}(\mathbf{p}(t))$ can be expressed as convex combination of system matrices $\mathbf{S}_r$. The explicit form of the TP model in terms of tensor product becomes:

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{pmatrix} \approx \mathcal{S} \bigotimes_{n=1}^{N} \mathbf{w}_n(p_n(t)) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix} \qquad (3)$$

that is

$$\left\| \mathbf{S}(\mathbf{p}(t)) - \mathcal{S} \bigotimes_{n=1}^{N} \mathbf{w}_n(p_n(t)) \right\| \le \varepsilon.$$

Here, ε symbolizes the approximation error, row vector $\mathbf{w}_n(p_n) \in \mathbb{R}^{I_n}$ $n = 1, \ldots, N$ contains the one variable weighting functions $w_{n,i_n}(p_n)$. Function $w_{n,j}(p_n(t)) \in [0,1]$ is the j-th one variable weighting function defined on the n-th dimension of Ω, and $p_n(t)$ is the n-th element of vector $\mathbf{p}(t)$. I_n $(n = 1, \ldots, N)$ is the number of the weighting functions used in the n-th dimension of the parameter vector $\mathbf{p}(t)$. The $(N+2)$-dimensional tensor $\mathcal{S} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_N\times O\times I}$ is constructed from LTI vertex systems $\mathbf{S}_{i_1 i_2 \ldots i_N} \in \mathbb{R}^{O\times I}$. For further details we refer to [2, 3, 4]. The convex combination of the LTI vertex systems is ensured by the conditions:

Definition 1 *The TP model (3) is convex if:*

$$\forall n \in [1,N], i, p_n(t) : w_{n,i}(p_n(t)) \in [0,1]; \quad (4)$$

$$\forall n \in [1,N], p_n(t) : \sum_{i=1}^{I_n} w_{n,i}(p_n(t)) = 1. \quad (5)$$

This simply means that $\mathbf{S}(\mathbf{p}(t))$ is within the convex hull of the LTI vertex systems $\mathbf{S}_{i_1 i_2 \ldots i_N}$ for any $\mathbf{p}(t) \in \Omega$.

$\mathbf{S}(\mathbf{p}(t))$ has a finite element TP model representation in many cases ($\varepsilon = 0$ in (3)). However, exact finite element TP model representation does not exist in general ($\varepsilon > 0$ in (3)), see Ref. [13]. In this case $\varepsilon \mapsto 0$, when the number of the LTI systems involved in the TP model goes to ∞. In this paper we will show that the LPV model of the aeroelastic system can be exactly represented by a finite TP model.

C. TP model transformation

The TP model transformation starts with the given LPV model (**??**) and results in the TP model representation (3), where the trade-off between the number of LTI vertex systems and the ε is optimized [3]. The TP model transformation offers options to generate different types of the weighting functions $w(\cdot)$. For instance:

Definition 2 ***SN - Sum Normalisation*** *Vector* $\mathbf{w}(p)$*, containing weighting functions* $w_i(p)$ *is SN if the sum of the weighting functions is 1 for all* $p \in \Omega$*.*

Definition 3 ***NN - Non Negativeness*** *Vector* $\mathbf{w}(p)$*, containing weighting functions* $w_i(p)$ *is NN if the value of the weighting functions is not negative for all* $p \in \Omega$*.*

Definition 4 ***NO - Normality*** *Vector* $\mathbf{w}(p)$*, containing weighting functions* $w_i(p)$ *is NO if it is SN and NN type, and the maximum values of the weighting functions are one. We say* $w_i(p)$ *is close to NO if it is SN and NN type, and the maximum values of the weighting functions are close to one.*

Definition 5 ***RNO - Relaxed Normality*** *Vector* $\mathbf{w}(p)$*, containing weighting functions* $w_i(p)$ *is RNO if the maximum values of the weighting functions are the same.*

Definition 6 ***INO - Inverted Normality*** *Vector* $\mathbf{w}(p)$*, containing weighting functions* $w_i(p)$ *is INO if the minimum values of the weighting functions are zero.*

All the above definitions of the weighting functions determine different types of convex hulls of the given LPV model. The SN and NN types guarantee (4), namely, they guarantee the convex hull. The TP model transformation is capable of always resulting SN and NN type weighting functions. This means that one can focus on applying LMI's developed for convex decompositions only, which considerably relaxes the further LMI design. The NO type determines a tight convex hull where as many of the LTI systems as possible are equal to the $\mathbf{S}(\mathbf{p})$ over some $\mathbf{p} \in \Omega$ and the rest of the LTI's are close to $\mathbf{S}(\mathbf{p}(t))$ (in the sense of L_2 norm). The SN, NN and RNO type guarantee that those LTI vertex systems which are not identical to $\mathbf{S}(\mathbf{p})$ are in the same distance from $\mathbf{S}(\mathbf{p}(t))$. INO guarantees that different subsets of the LTI's define $\mathbf{S}(\mathbf{p}(t))$ over different regions of $\mathbf{p} \in \Omega$.

These different types of convex hulls strongly effect the feasibility of the further LMI design. For instance paper [1] shows an example when determining NO is useful in the case of controller design while the observer design is more advantageous in the case of INO type weighting functions.

In order to have a direct link between the TP model form and the typical form of LMI conditions, we define the following index transformation:

Definition 7 *(*Index transformation) *Let*

$$\mathbf{S}_r = \begin{pmatrix} \mathbf{A}_r & \mathbf{B}_r \\ \mathbf{C}_r & \mathbf{D}_r \end{pmatrix} = \mathbf{S}_{i_1, i_2, .., i_N},$$

where $r = ordering(i_1, i_2, .., i_N)$ *(*$r = 1..R = \prod_n I_n$*). The function "ordering" results in the linear index equivalent of an N dimensional array's index* $i_1, i_2, .., i_N$*, when the size of the array is* $I_1 \times I_2 \times .. \times I_N$*. Let the weighting functions be defined according to the sequence of* r*:*

$$w_r(\mathbf{p}(t)) = \prod_n w_{n,i_n}(p_n(t)).$$

By the above index transformation one can write the TP model (3) in the typical form of:

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{R} w_r(\mathbf{p}(t)) \mathbf{S}_r.$$

Note that the LTI systems $\mathbf{S}_r$ and $\mathbf{S}_{i_1, i_2, .., i_N}$ are the same, only their indices are modified, therefore the hull defined by the LTI systems is the same in both forms.

III. CASE STUDY OF THE PROTOTYPICAL AEROELASTIC WING SECTION

The prototypical aeroelastic wing section is used for the theoretical as well as experimental analysis of two- dimensional aeroelastic behavior. It has complex dynamic behavior. One can find a whole series of detailed studies of this wing section in the *Journal of Guidance, Control and Dynamic.* For more details we refer to [2, 1].

Let us consider the problem of flutter suppression for the prototypical aeroelastic wing section as shown in Figure 1. The flat plate airfoil is constrained to have two degrees of freedom, the plunge h and pitch α. In order to have a deep description of the equations of motion, we refer to Refs. [6, 7, 9, 10, 14]. Here we give only a brief discussion. The equations of motion in linear parameter-varying state-space form is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t) = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}, \quad (6)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{pmatrix} \quad \text{and} \quad \mathbf{u}(t) = \beta$$

and

$$\mathbf{A}(\mathbf{p}(t)) =$$

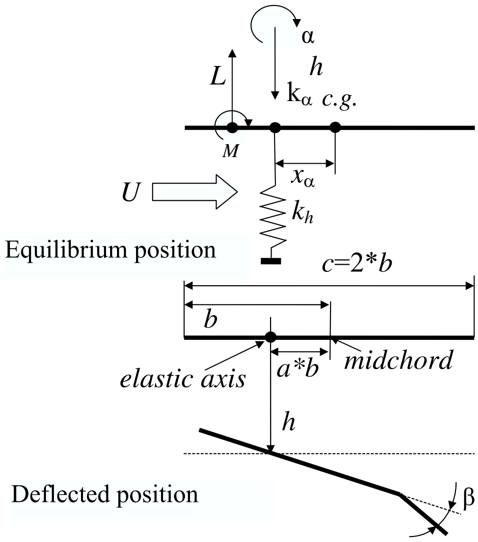

Fig. 1: Two-dimensional flat plate airfoil small deflection, force notation and schematic diagram

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -(k_2U^2+p(x_2(t))) & -c_1(U) & -c_2(U) \\ -k_3 & -(k_4U^2+q(x_2(t))) & -c_3(U) & -c_4(U) \end{pmatrix},$$

$$\mathbf{B}(\mathbf{p}(t)) = \begin{pmatrix} 0 \\ 0 \\ g_3U^2 \\ g_4U^2 \end{pmatrix},$$

where $\mathbf{p}(t) \in \mathbb{R}^{N=2}$ contains values $x_2(t) = \alpha$ and U. Further $d = m(I_\alpha - mx_\alpha^2 b^2)$;

$k_1 = \frac{I_\alpha k_h}{d}$; $k_2 = \frac{I_\alpha \rho b c_{l_\alpha} + mx_\alpha b^3 \rho c_{m_\alpha}}{d}$;

$k_3 = \frac{-mx_\alpha b k_h}{d}$; $k_4 = \frac{-mx_\alpha b^2 \rho c_{l_\alpha} - m\rho b^2 c_{m_\alpha}}{d}$;

$p(\alpha) = \frac{-mx_\alpha b}{d} k_\alpha(\alpha)$; $q(\alpha) = \frac{m}{d} k_\alpha(\alpha)$;

$c_1(U) = \left(I_\alpha(c_h + \rho U b c_{l_\alpha}) + mx_\alpha \rho U^3 c_{m_\alpha}\right)/d$;

$c_2(U) = \left(I_\alpha \rho U b^2 c_{l_\alpha}(\frac{1}{2} - a) - mx_\alpha b c_\alpha + mx_\alpha \rho U b^4 c_{m_\alpha}(\frac{1}{2} - a)\right)/d$;

$c_3(U) = \left(-mx_\alpha b c_h - mx_\alpha \rho U b^2 c_{l_\alpha} - m\rho U b^2 c_{m_\alpha}\right)/d$;

$c_4(U) = \left(mc_\alpha - mx_\alpha \rho U b^3 c_{l_\alpha}(\frac{1}{2} - a) - m\rho U b^3 c_{m_\alpha}(\frac{1}{2} - a)\right)/d$;

$g_3 = (-I_\alpha \rho b c_{l_\beta} - mx_\alpha b^3 \rho c_{m_\beta})/d$;

$g_4 = (mx_\alpha b^2 \rho c_{l_\beta} + m\rho b^2 c_{m_\beta})/d$;

The system parameters are given in the Appendix. These data are obtained from experimental models described in full detail in Refs. [9, 11].

$$k_\alpha(\alpha) = 2.82(1 - 22.1\alpha + 1315.5\alpha^2 + 8580\alpha^3 + 17289.7\alpha^4)$$

is obtained by curve fitting on the measured displacement-moment data for non-linear spring [11]. We remark that the uncontrolled response of the system achieves limit cycle oscillation as claimed in Refs. [9, 11, 15]. One should note that the equations of motion are also dependent on the elastic axis location a.

A. *TP model representations of the prototypical aeroelastic wing section*

This subsection presents different TP model representations of the LPV model (6). We execute the TP model transformation over a $M_1 \times M_2$, ($M_1 = 101$ and $M_2 = 101$) hyper grid net in $\Omega : [14, 25] \times [-0.1, 0.1]$ ($U \in [14, 25](m/s)$ and $\alpha \in [-0.1, 0.1](rad)$). The TP model transformation shows that the LPV model of the wing section can exactly be given by TP model with 6 LTI vertex models, namely, by the parameter varying convex combination of 6 LTI models:

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{i=1}^{3} \sum_{j=1}^{2} w_{1,i}(U(t)) w_{2,j}(\alpha(t)) \mathbf{S}_{i,j}$$

In the followings we show that the type of the convex combination can readily be modified by the TP model transformation:

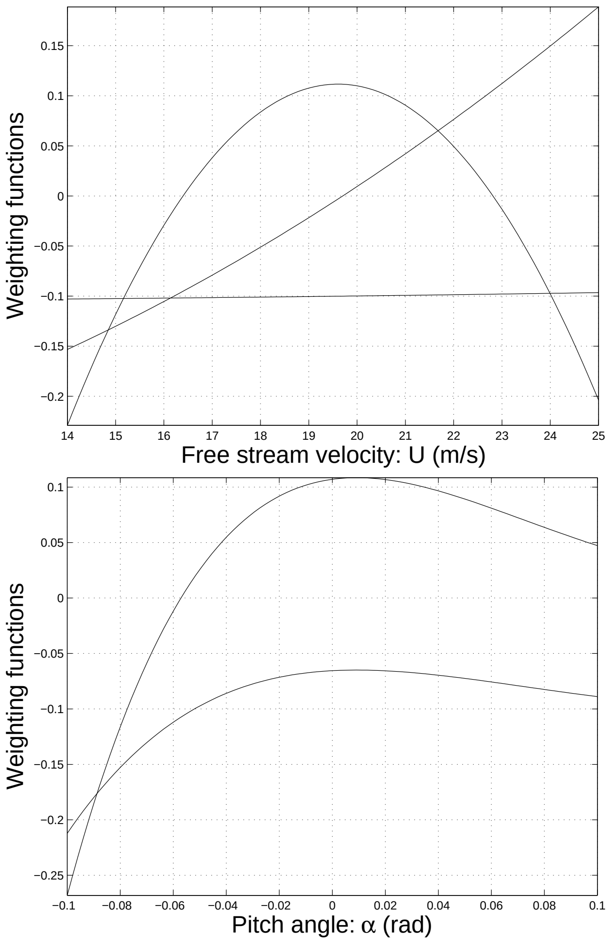

Fig. 2: Weighting functions of the TP model 0 on the dimensions α and U.

TP MODEL 0: The resulting weighting functions depicted on Figure 2 are directly obtained by the TP model transformation without any further modification. They are between -1 and $+1$ and orthogonal. The resulting LTI vertex systems do not define the convex hull of the LPV model, but their number is minimized.

TP MODEL 1: In order to have convex TP model to which the LMI control design conditions can be applied, let us generate SN and NN type weighting functions by the TP model transformation. The results are depicted on Figure 3.

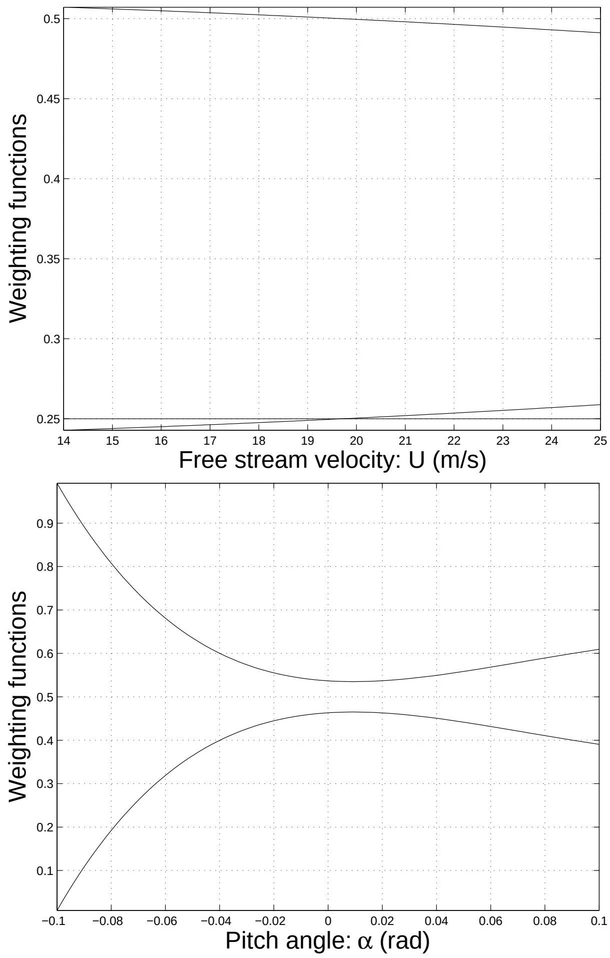

Fig. 3: SN and NN type weighting functions of the TP model 1 on the dimensions α and U.

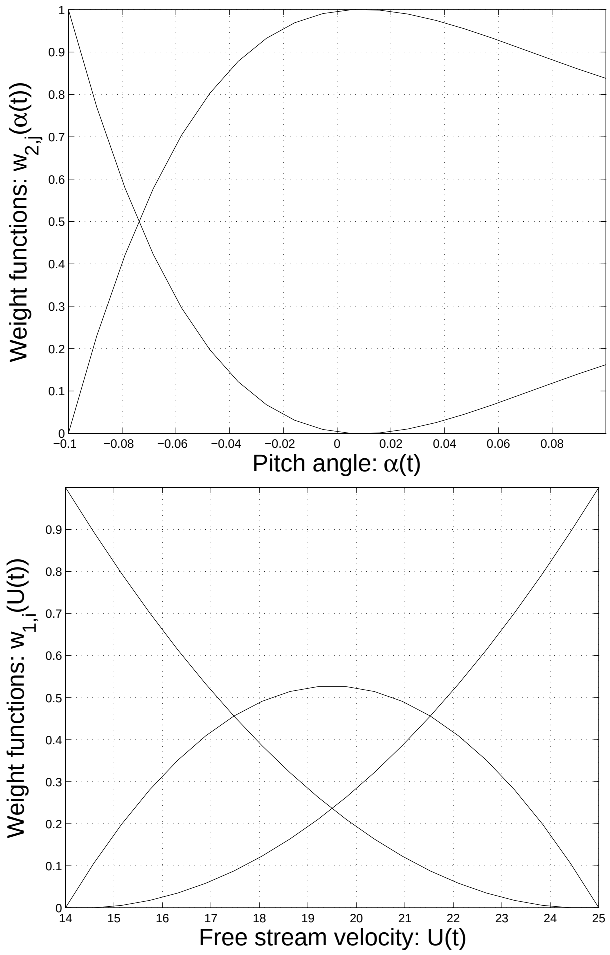

Fig. 4: Close to NO type weighting functions of the TP model 1 on the dimensions α and U.

TP MODEL 2: In many cases the convexity of the TP model is not enough, the further LMI design is not feasible. In order to relax the feasibility of the LMI conditions, let us define the tight convex hull of the LPV model via generating close to NO type weighting functions by the TP model transformation, see Figure 7.

TP MODEL 3: Let us further modify the weighting functions and define their INO - RNO type, see Figure 5. Paper [1] shows that this type is advantageous in the case of observer design.

Perhaps the above resulting weighting functions can be derived analytically. The functions $w(\alpha)$ can be derived from k_α. The analytical derivation of $w(U)$, however, seems to be rather complicated. The analytical derivations of the tight convex hull or INO - RNO type weighting functions need the analytical solution of the tight convex hull problem that is unavailable in general. In spite of this, the TP model transformation requires a few minutes and is not dependent on the actual analytical form of the given LPV model. If the model is changed we can simply execute the TP model transformation again.

IV. TYPICAL AFFINE MODEL FORM

TP model 2 was applied in [2] to design stabilizing controller. Let us transform TP model 2

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{i=1}^{3}\sum_{j=1}^{2} w_{1,i}(U(t))w_{2,j}(\alpha(t))\mathbf{S}_{i,j}$$

to the typical affine model form:

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{6} w_r(U(t),\alpha(t))\mathbf{S}_r,$$

where $\mathbf{S}_r = \mathbf{S}_{i,j}$, $w_r(U(t),\alpha(t)) = w_{1,i}(U(t))w_{2,j}(\alpha(t))$ and $r = 2(i-1)+j$ (see Definition 7).

The weighting functions $w_r(\cdot)$ are presented on the Figures 6 and 7.

V. CONCLUSION

This paper shows how the TP model transformation is capable of defining affine models with various types of convex hulls of a given LPV model in a few minutes without analytical derivations. We may conclude that the TP model may replace the analytic affine model decomposition. We studied the example of the LPV model of the prototypical aeroelastic wing section.

VI. APPENDIX

Nomenclature

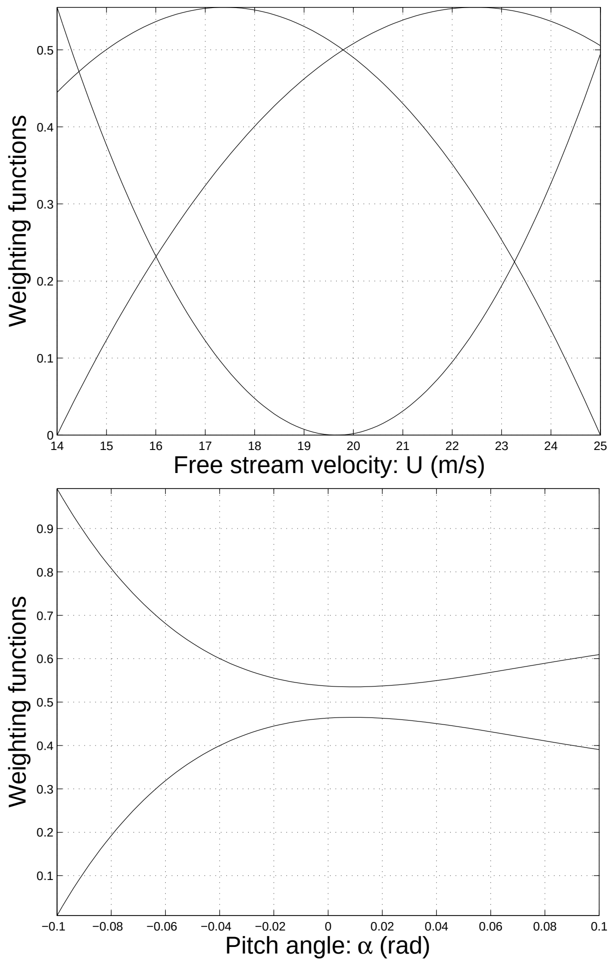

Fig. 5: INO-RNO type weighting functions of the TP model 2 on the dimensions α and U.

- h = plunging displacement
- α = pitching displacement
- x_α = the non-dimensional distance between elastic axis and the center of mass
- m = the mass of the wing
- I_α = the mass moment of inertia
- b = semi-chord of the wing
- c_α = the pitch structural damping coefficient
- c_h = the plunge structural damping coefficient
- k_h = the plunge structural spring constant
- $k_\alpha(\alpha)$ = non-linear stiffness contribution
- L = aerodynamic force
- M = aerodynamic moment
- β = control surface deflection
- ρ = air density
- U = free stream velocity
- c_{l_α} = lift coefficients per angle of attack

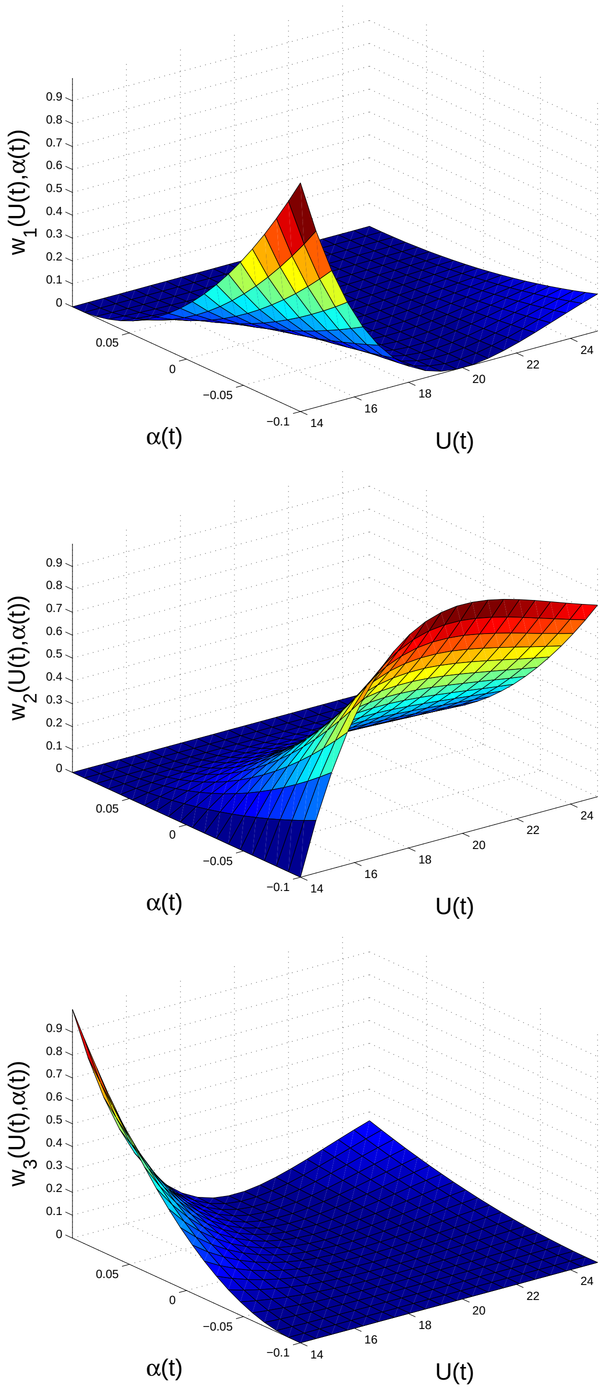

Fig. 6: Weighting functions of the affine model

- c_{m_α} = moment coefficients per angle of attack
- c_{l_β} = lift coefficients per control surface deflection
- c_{m_β} = moment coefficients per control surface deflection
- a = non-dimensional distance from the midchord to the elastic axis

System parameters

$b = 0.135m$; $span = 0.6m$; $k_h = 2844.4N/m$; $c_h = 27.43Ns/m$; $c_\alpha = 0.036Ns$; $\rho = 1.225kg/m^3$; $c_{l_\alpha} = 6.28$;

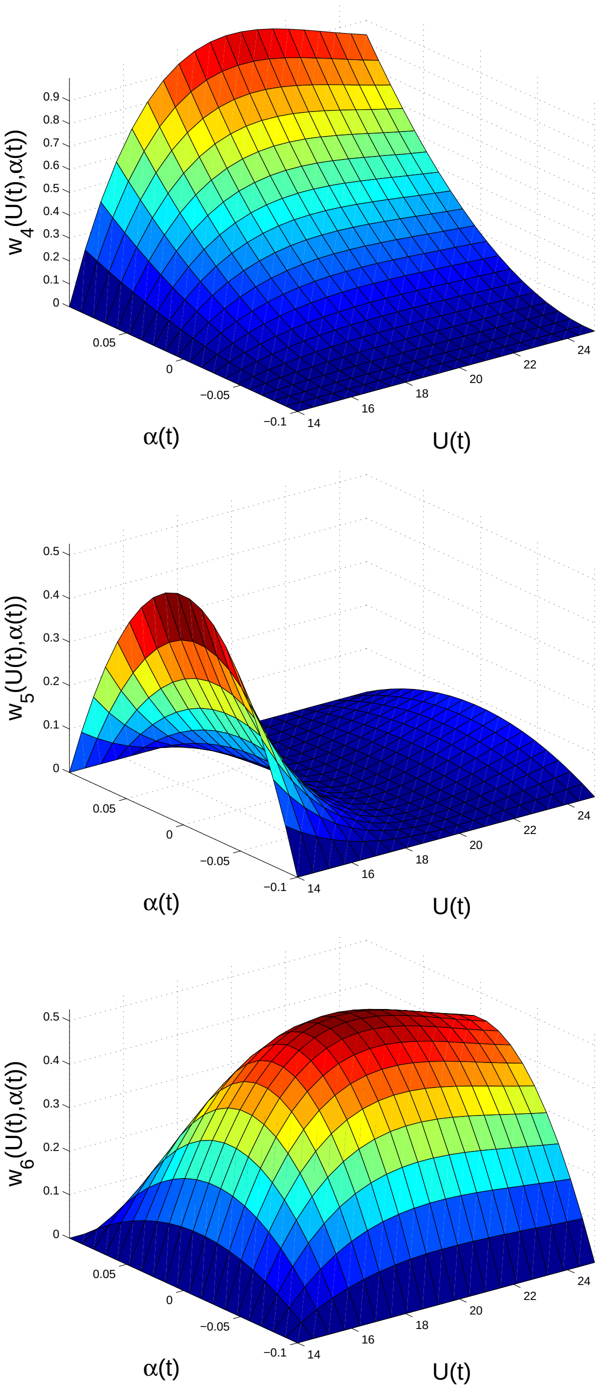

Fig. 7: Weighting functions of the affine model

$c_{l_\beta} = 3.358$; $c_{m_\alpha} = (0.5 + a)c_{l_\alpha}$; $c_{m_\beta} = -0.635$; $m = 12.387kg$; $x_\alpha = -0.3533 - a$; $I_\alpha = 0.065kgm^2$; $c_\alpha = 0.036$;

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