Different Affine Decomposition of the Model of the TORA System by TP model transformation

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***Abstract* — The Tensor Product (TP) model transformation is a recently proposed technique for transforming given Linear Parameter Varying (LPV) models into affine model form, namely, to parameter varying convex combination of Linear Time Invariant (LTI) models. The main advantage of the TP model transformation is that the Linear Matrix Inequality (LMI) based control design frameworks can immediately be applied to the resulting affine models to yield controllers with tractable and guaranteed performance. The effectiveness of the LMI design depends on the LTI models of the convex combination. Therefore, the main objective of this paper is to study how the TP model transformation is capable of determining different types of convex hulls of the LTI models. This paper shows a case study of the TORA system. The theory and the definitions of the affine decomposition is discussed in the paper "Different Affine Decomposition of the Model of the Prototypical Aeroelastic Wing Section by TP model transformation, PART I" of this proceedings.**

I CASE STUDY OF THE TORA SYSTEM

The Translational Oscillations with a Rotational Actuator (TORA) system[1] was developed as a simplified model of a dual-spin spacecraft [13]. Later, Bernstein and his colleagues at the University of Michigan, Ann Arbor, turned it into a benchmark problem for nonlinear control [1, 2, 3].

The system shown in Fig. 1 represents a translational oscillator with an eccentric rotational proof-mass actuator. The oscillator consists of a cart of mass M connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel. The proof-mass actuator attached to the cart has mass m and moment of inertia I about its center of mass, which is located at distance e from the point about which the proof mass rotates. The motion occurs in a horizontal plane, so that no gravitational forces need to be considered. In Fig. 1, N denotes the control torque applied to the proof mass, and F is the disturbance force on the cart.

[1]Also referred to as the rotational/translational proof-mass actuator (RTAC) system.

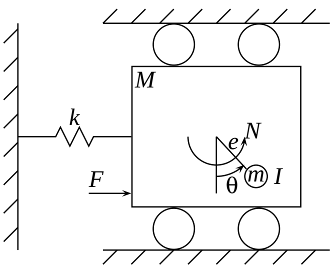

Fig. 1: TORA system

Let q and $\dot{q}$ denote the translational position and velocity of the cart, and let θ and $\dot{\theta}$ denote the angular position and velocity of the rotational proof mass, where $\theta = 0\,\mathrm{deg}$ is perpendicular to the motion of the cart, and $\theta = 90\,\mathrm{deg}$ is aligned with the positive q direction. The equations of motion are given by

$$\begin{aligned}(M+m)\ddot{q}+kq &= -me(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+F\\ (I+me^2)\ddot{\theta} &= -me\ddot{q}\cos\theta+N\end{aligned}$$

With the normalization

$$\xi \triangleq \sqrt{\tfrac{M+m}{I+me^2}}q, \qquad \tau \triangleq \sqrt{\tfrac{k}{M+m}}t,$$
$$u \triangleq \tfrac{M+m}{k(I+me^2)}N, \quad w \triangleq \tfrac{1}{k}\sqrt{\tfrac{M+m}{I+me^2}}F,$$

the equation of motion become

$$\begin{aligned}\ddot{\xi}+\xi &= \varepsilon\left(\dot{\theta}^2\sin\theta-\ddot{\theta}\cos\theta\right)+w\\ \ddot{\theta} &= -\varepsilon\ddot{\xi}\cos\theta+u\end{aligned}$$

where ξ is the normalized cart position, and w and u represent the dimensionless disturbance and control torque, respectively. In the normalized equations, the symbol $(\cdot)$ represents differentiation with respect to the normalized time τ. The coupling between the translational and rotational motions is represented by the parameter ε which is defined by

$$\varepsilon \triangleq \frac{me}{\sqrt{(I+me^2)(M+m)}}$$

Letting $\mathbf{x} = \begin{pmatrix}x_1 & x_2 & x_3 & x_4\end{pmatrix}^T = \begin{pmatrix}\xi & \dot{\xi} & \theta & \dot{\theta}\end{pmatrix}^T$, the dimensionless equations of motion in first-order form are given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})u+\mathbf{d}(\mathbf{x})w, \tag{1}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix}0 & 1 & 0 & 0\\ \frac{-1}{1-\varepsilon^2\cos^2 x_3} & 0 & 0 & \frac{\varepsilon x_4\sin x_3}{1-\varepsilon^2\cos^2 x_3}\\ 0 & 0 & 0 & 1\\ \frac{\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3} & 0 & 0 & \frac{-\varepsilon x_4\sin x_3}{1-\varepsilon^2\cos^2 x_3}\end{pmatrix},$$

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix}0\\ \frac{-\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3}\\ 0\\ \frac{1}{1-\varepsilon^2\cos^2 x_3}\end{pmatrix}, \qquad \mathbf{d}(\mathbf{x}) = \begin{pmatrix}0\\ \frac{1}{1-\varepsilon^2\cos^2 x_3}\\ 0\\ \frac{-\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3}\end{pmatrix},$$

Table 1: Parameters of the TORA system

Description	Parameter	Value	Units
Cart mass	M	1.3608	kg
Arm mass	m	0.096	kg
Arm eccentricity	e	0.0592	m
Arm inertia	I	0.0002175	kg m^2
Spring stiffness	k	186.3	N/m
Coupling parameter	ε	0.200	—

Note that u, the control input, is the normalized torque N and w, the disturbance, is the normalized force F. In the followings consider the case of no disturbance. The parameters of the simulated system are given in Table 1.

A Determination of the convex state-space TP model form of the TORA system

Observe that the nonlinearity is caused by $x_3(t)$ and $x_4(t)$. For the TP model transformation we define the transformation space as $\Omega = [-a,a]\times[-a,a]$ ($x_3(t)\in[-a,a]$ and $x_4(t)\in[-a,a]$), where $a=\frac{45}{180}\pi$rad (note that these intervals can be arbitrarily defined). Let the density of the sampling grid be 101×101. The sampling results in $\mathbf{A}^s_{i,j}$ and $\mathbf{B}^s_{i,j}$, where $i,j=1\dots 101$. Then we construct the matrix $\mathbf{S}^s_{i,j} = \begin{pmatrix}\mathbf{A}^s_{i,j} & \mathbf{B}^s_{i,j}\end{pmatrix}$, and after that the tensor $\mathcal{S}^s\in\mathbb{R}^{101\times101\times4\times4}$ from $\mathbf{S}^s_{i,j}$. If we execute HOSVD on the first two dimensions of $\mathcal{S}^s$ then we find that the rank of $\mathcal{S}^s$ on the first two dimensions are 4 and 2 respectively. This means that the TORA system can be exactly given as convex combination of $4\times2=8$ linear vertex model. The TP model transformation describes TORA system as:

$$\dot{\mathbf{x}}(t)=\sum_{i=1}^{4}\sum_{j=1}^{2} w_{1,i}(x_3(t))w_{2,j}(x_4(t))\left(\mathbf{A}_{i,j}\mathbf{x}(t)+\mathbf{B}_{i,j}u(t)\right). \tag{2}$$

In the followings we show that the type of the convex combination can readily be modified by the TP model transformation:

TP MODEL 0: The resulting weighting functions depicted on Figure 2 are directly obtained by the TP model transformation without any further modification. They are between -1 and $+1$ and orthogonal. The resulting LTI vertex systems do not define the convex hull of the LPV model, but their number is minimized.

TP MODEL 1: In order to have convex TP model to which the LMI control design conditions can be applied, let us generate SN and NN type weighting functions by the TP model transformation. The results are depicted on Figure 3.

TP MODEL 2: In many cases the convexity of the TP model is not enough, the further LMI design is not feasible. In order to relax the feasibility of the LMI conditions, let us define the tight convex hull of the LPV model via generating close to NO type weighting functions by the TP model transformation, see Figure 4.

TP MODEL 3: Let us further modify the weighting functions and define their INO–RNO type, see Figure 5.

The above resulting weighting functions can be derived analytically in some cases, but as the model become more and more complex, the analytical derivations needs more and more expertise. Moreover, the analytical derivations of

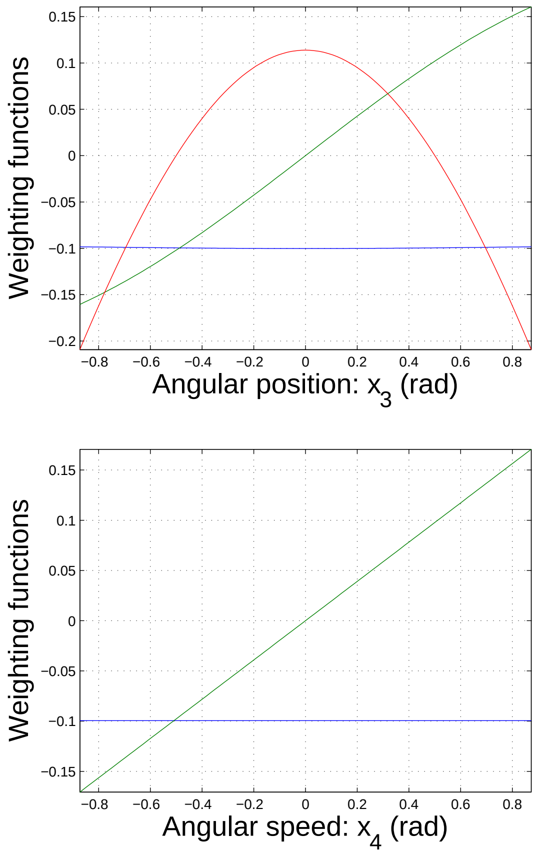

Fig. 2: Weighting functions of the TP model 0 on dimensions $x_3(t)$ and $x_4(t)$

the tight convex hull or INO–RNO type weighting functions need the analytical solution of the tight convex hull problem that is unavailable in general. In spite of this, the TP model transformation requires a few minutes and is not dependent on the actual analytical form of the given LPV model. If the model is changed we can simply execute the TP model transformation again.

II TYPICAL AFFINE MODEL FORM

The TP model form can be transformed to the typical affine model form that can be used in other control theories. Let us transform the TP model

$$\mathbf{S}(\mathbf{p}(t))=\sum_{i=1}^{4}\sum_{j=1}^{2} w_{1,i}(x_3(t))w_{2,j}(x_4(t))\mathbf{S}_{i,j}$$

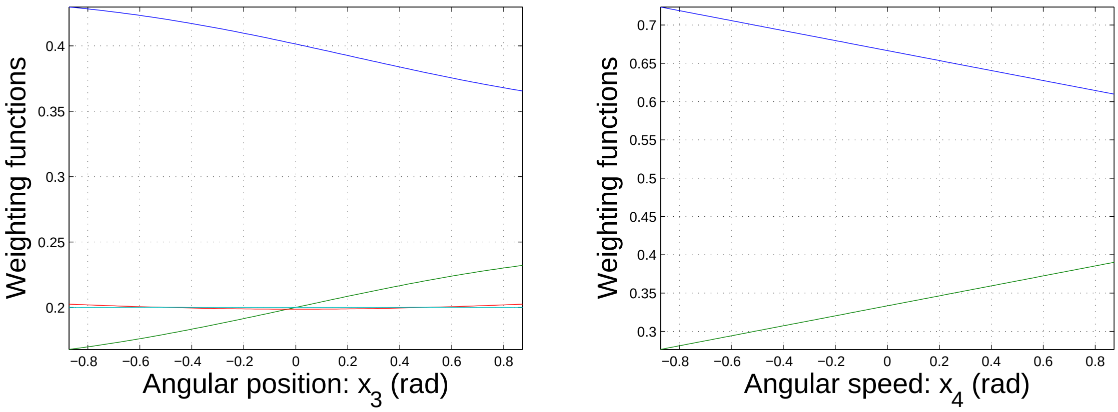

Fig. 3: SN and NN type weighting functions of the TP model 1 on dimensions $x_3(t)$ and $x_4(t)$

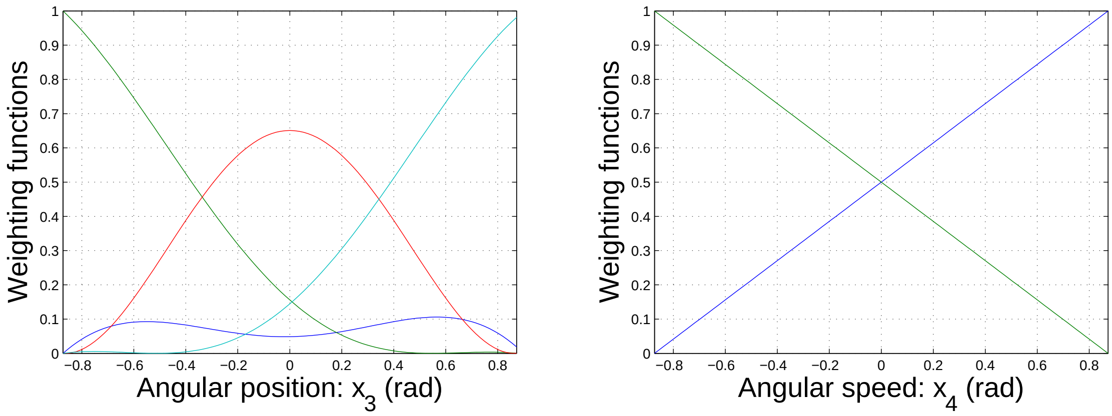

Fig. 4: Close to NO type weighting functions of the TP model 2 on dimensions $x_3(t)$ and $x_4(t)$

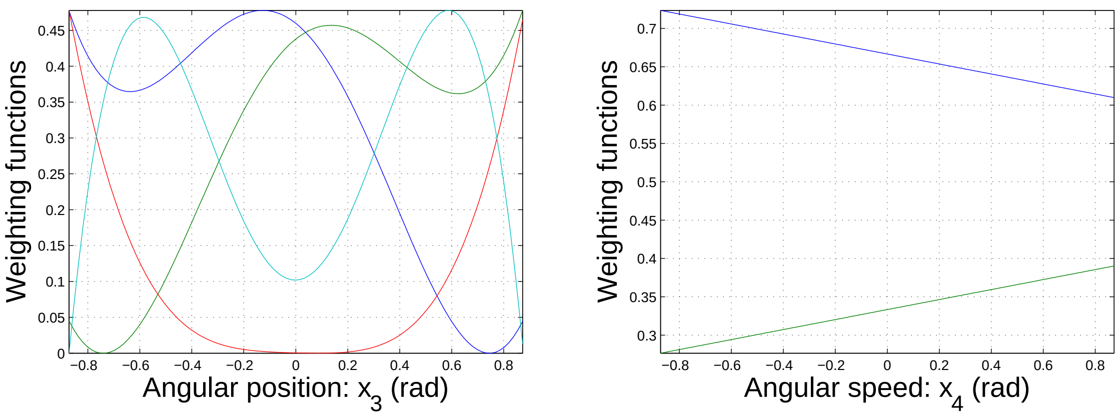

Fig. 5: INO–RNO type weighting functions of the TP model 3 on dimensions $x_3(t)$ and $x_4(t)$

to the typical affine model form:

$$\mathbf{S}(\mathbf{p}(t)) = \sum_{r=1}^{8} w_r(x_3(t), x_4(t))\mathbf{S}_r,$$

where $\mathbf{S}_r = \mathbf{S}_{i,j}$, $w_r(x_3(t), x_4(t)) = w_{1,i}(x_3(t))w_{2,j}(x_4(t))$ and $r = 2(i-1)+j$.

The weighting functions $w_r(\cdot)$ of TP model 0 in the typical affine model from are presented in Fig. 6. As a comparison, the weighting functions $w_r(\cdot)$ of TP model 2 are also depicted in Fig. 7. As the paper [4] presents, the stabilizing controller derived from TP model 2 affine model showed better control performance than TP model 1.

III CONCLUSION

This paper shows how the TP model transformation is capable of defining affine models with various types of convex hulls of a given LPV model in a few minutes without analytical derivations. We may conclude that the TP model may replace the analytic affine model decomposition. The paper

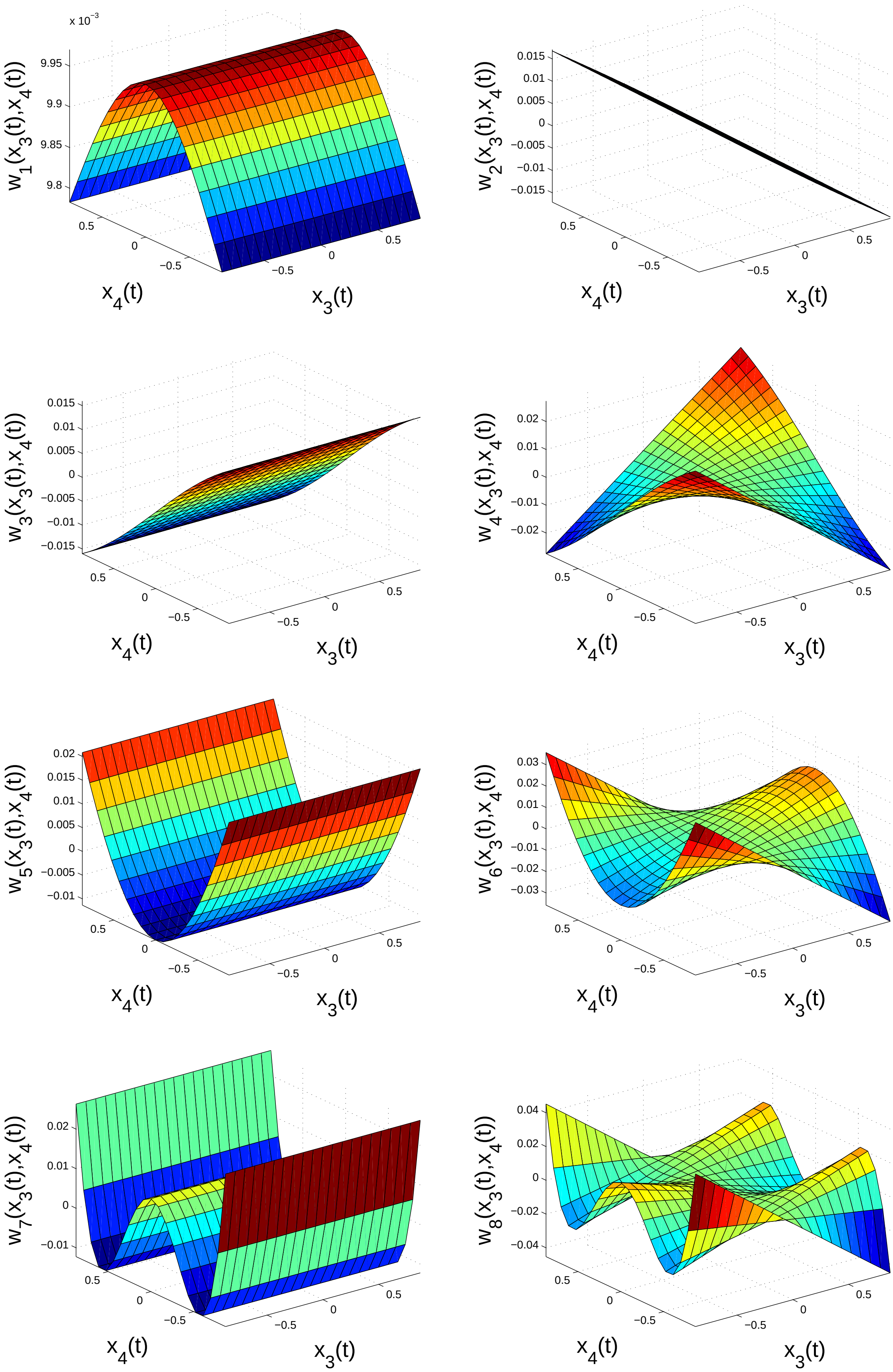

Fig. 6: Weighting functions of the affine model

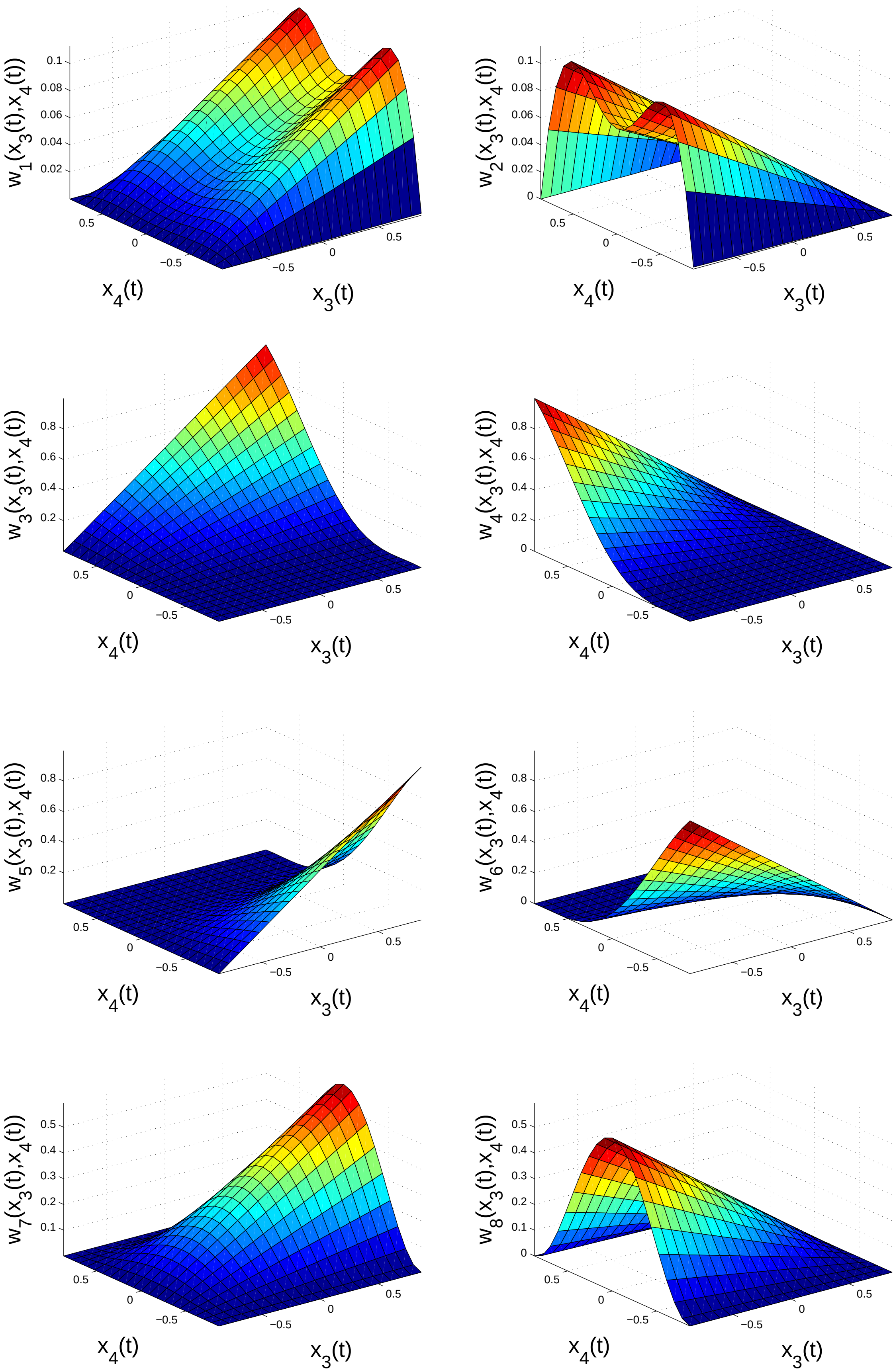

Fig. 7: Close to NO type weighting functions of the affine model

represented the different affine models for the example of the LPV model of the TORA system.

IV ACKNOWLEDGEMENT

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