PARALLEL IMPLEMENTATION OF THE RECURSIVE DISCRETE FOURIER TRANSFORMATION[1]

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Abstract **- In this paper a new composite filter-bank structure is presented for the efficient implementation of the recursive discrete Fourier transformation and some of its generalizations. This structure is based on a proper combination of the concepts of polyphase filtering and the fast Fourier transformation (FFT) algorithm. Its computational complexity is in direct correspondence with the FFT, and can be operated both in sliding and block-oriented modes. Based on this filter-bank almost all issues of the so-called transform-domain digital signal processing can be reconsidered. The inherent parallelism of this structure enables very high speed in practical implementations.**

I. INTRODUCTION

The recursive Discrete Fourier Transformation (DFT) is a sliding-window technique for calculating the Fourier coefficients and/or components with a computational complexity proportional to the window size for every input sample [1]. The classical version based on the Lagrange structure has been replaced by an observer structure [2,3] having very good properties [4,5]. This latter consists of parallel resonators in a common feedback loop, and operates as a single-input multiple-output (SIMO) filter-bank.

This paper presents a new technique for the implementation of the recursive DFT. The novelty of this approach is a decomposition of a larger size (SIMO) DFT filter-bank into a proper parallel combination of smaller ones. On the input side the decomposition follows the "decimation-in-time" (see e.g. [6]) principle, and results in a set of smaller size recursive DFT blocks operating at lower sampling frequency in a polyphase form [7]. A proper scanning of this polyphase form provides the input sequences for the recursive DFT blocks of the output stage. These DFT blocks follow the "decimation-in-frequency" principle, and produce the Fourier coefficients and/or components of the larger size DFT. Obviously for every smaller size recursive DFT block a similar decomposition is possible if its size isn't a prime number. A "total" decomposition corresponds to the structure of the Fast Fourier Transform (FFT), but unlike the FFT operates recursively.

The real novelty of the proposed algorithm with respect to other techniques of similar complexity [8,9] is its polyphase nature providing additional advantages in parallelization, and its ability to implement recursive DFT with fading memory [10].

II. NOTATIONS

N	-the size of the DFT,
M	-the size of the input-stage DFTs
L	-number of the input-stage DFTs ($N=ML$)
x_n	-nth input sample, n=0,1,...,N-1, or running,
n	-discrete "time" index,
X_m	-mth DFT coefficient, m=0,1,...,N-1,
m	-discrete "frequency" index,
l	-identifier index for blocks of size M, l=0,1,...,L-1,
k	-running index within blocks of size M, k=0,1,...,M-1.

III. DEVELOPMENT OF THE ALGORITHM

The DFT of a sequence of N samples is defined as

$$X_m = \frac{1}{N}\sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}mn}, \tag{1}$$

where the factor of $1/N$ is a simple normalizing factor used typically with the inverse transform (see e.g. [5]), but preferred in (1) in the case of recursive implementation. As a first step let's decompose the input sequence of N samples into L blocks of size M following the "decimation in time" principle:

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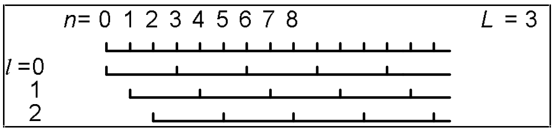

With this decomposition (1) can be rewritten as

$$X_m = \frac{1}{N}\sum_{k=0}^{M-1} x_{kL} e^{-j\frac{2\pi}{N}mkL} + \frac{1}{N}\sum_{k=0}^{M-1} x_{kL+1} e^{-j\frac{2\pi}{N}m(kL+1)} + \ldots$$
$$+ \frac{1}{N}\sum_{k=0}^{M-1} x_{kL+L-1} e^{-j\frac{2\pi}{N}m(kL+L-1)} = \frac{1}{L}\sum_{l=0}^{L-1}\mathrm{DFT_M}(m,l)\, e^{-j\frac{2\pi}{N}ml}, \tag{2}$$

where $\mathrm{DFT}_M(m,l)$ stands for the mth "frequency" component of the lth M-size DFT. From this decomposition it is clear that a DFT of size N can be evaluated as L separate DFTs of size M together with a "phase compensation" according to (2).

This decomposition can be utilized also for the recursive DFT. The recursive DFT structures are filter-banks producing Fourier coefficients and/or components. In every step the structures with component outputs provide the DFT of the last N input samples, while in the case of coefficient outputs the periodically "phase shifted" DFT (see e.g. [11]) is produced. The proposed decomposition can be directly applied for both cases, but for simplicity, in this paper, only the coefficient output version is considered in details. The input sequence to be transformed is fed to the input of the L separate recursive DFT filter-banks corresponding to the above figure. The parallel output channels of each filter-bank produce a new output sample in every Lth time instant. Due to the polyphase nature of operation, the evaluation of (2) can be performed only after collecting of the L subsequent samples from the (m modulo M)th channels of the separate recursive DFT units. These samples will appear at the rate of the original sampling frequency, which means that a recursive evaluation of (2) would also be of considerable interest. By applying the "decimation-in-frequency" principle a new set of recursive DFTs gives the solution, since (2) can be rewritten as

$$X_m = \frac{1}{L}\sum_{l=0}^{L-1}\mathrm{DFT_M}(m,l)\, e^{-j\frac{2\pi}{N}ml} = \frac{1}{L}\sum_{l=0}^{L-1}\mathrm{DFT_M}(m,l)\, e^{-j\frac{2\pi}{LM}ml}, \tag{3}$$

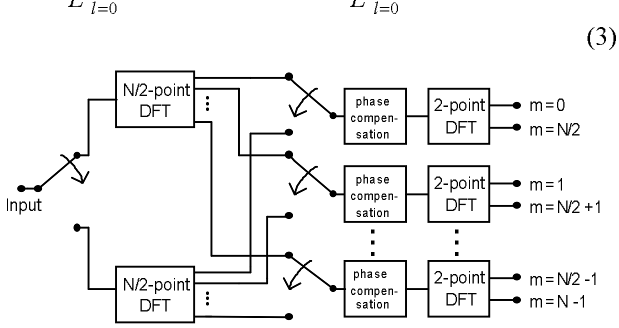

Fig. 1 Polyphase decomposition for M=N/2, and L=2.

from which

X_m= $\mathrm{DFT_L}\{\mathrm{DFT_M}(m,l)\}$,
if m=0, M,...,(L-1)M
$\mathrm{DFT_L}\{\mathrm{DFT_M}(m,l)\exp[-j(2\pi/N)l]\}$,
if m=1, M+1,...,(L-1)M+1
•
•
•
$\mathrm{DFT_L}\{\mathrm{DFT_M}(m,l)\exp[-j(2\pi/N)(M-1)l]\}$,
if m=M-1, 2M-1, ...,LM-1

This means, that the evaluation of (2) requires a "phase compensation", and the operation of M DFT filter-banks of size L. Fig. 1 shows the proposed decomposition for M=N/2, and L=2. In this Figure the 2-point DFT blocks are preceded by a "phase compensation" which is unavoidable due to the delayed reference positions of the N/2-point DFT blocks. Since in general this "phase compensation" is a multiplication by a complex number and even the outputs of the previous DFT stages are complex, the 2-point DFT blocks operate on complex values, i.e. on real and imaginary parts, and correspondingly produce complex values at their outputs.

If the numbers M and/or L are not prime the corresponding DFT filter-banks can also be decomposed similar manner as it is made in the the case of FFT (see e.g. [5]).

The decomposition described above can be directly utilized even in the case of DFT filter-banks producing the Fourier components instead of the Fourier coefficients. For many applications this fact has real practical advantages. With a slightly different "phase compensation", i.e. with the proper combination of the complex demodulation-modulation and the phase compensation in (3), the DFT filter-bank version can also be generated. For this case the corresponding 2*2 DFT filter-bank forms are given in Fig. 2. Since these banks are simple 2-channel filters operating at "zero frequency" and at "one half of the sampling frequency", the overall structure represents a combination of filters and complex modulators: these latters are for positioning the frequency content in an appropriate way.

IV. RECURSIVE DFT WITH FADING MEMORY

As it is well-known from the literature (see e.g. [1]) the averaging effect in (1) can have an interesting frequency-domain interpretation. One of the typical characterizations is the z-domain transfer function of the averager

$$\frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}}. \tag{4}$$

This transfer function is present in every sliding DFT channel that produces the linear average of the last N (properly demodulated) input samples. A very similar characterization is valid for the DFT filter-bank

$$\frac{1}{N}\frac{1-z^{-N}}{1-z_m z^{-1}}, \quad z_m = e^{j\frac{2\pi}{N}m}, \quad m = 0,1,\ldots,N-1, \qquad (5)$$

which operates as an "averager" at the mth frequency position. Both (4) and (5) are FIR filters even if there are recursive implementations [3-5]. The extension of the recursive DFT to a fading memory version, as it is motivated in [10], can be easily solved with the application of recursive building blocks producing also certain poles for the overall transfer function. In the radix-2 case almost every kind of second-order IIR filter sections is acceptable, however, it is not the case for higher- order blocks. Here the resonator-based approach is applied, which proved to be advantageous from implementational point of view. Fig. 2 presents also a slightly modified version of the 2*2 DFT filter-bank. Here instead of the common multiplier 1/2 two new parameters are introduced: r_0, and r_1. With these the transfer function of the channels (only the pipelineable version is considered):

$$H_0(z) = \frac{r_0 z^{-1}(1+z^{-1})}{1+(r_0-r_1)z^{-1}+(r_0+r_1-1)z^{-2}}, \qquad (6)$$

$$H_1(z) = \frac{-r_1 z^{-1}(1-z^{-1})}{1+(r_0-r_1)z^{-1}+(r_0+r_1-1)z^{-2}}, \qquad (7)$$

where for $r_0 = r_1 = 1/2$ we have the standard (pipelined) solution, however, with any other combination of these parameters we will have poles outside the origin. The fading memory effect desribed in [10] requires uniformly distributed poles within the unit circle. The characterization of the modified DFT systems is given if (4) is replaced by:

$$\frac{1-a}{N}\frac{1-z^{-N}}{1-az^{-N}}\frac{1}{1-z^{-1}}, \qquad (8)$$

from which it turns out that the poles are the Nth roots of a. If we consider the behaviour of the polyphase structures [7], it turns out that these poles can be generated for the radix-2 case in such a way that the first stage of the overall structure consisting of $N/2$ 2*2 DFT blocks is realized using blocks of Fig. 2 with

$$r_0 = r_1 = \frac{1-a}{2} \qquad (9)$$

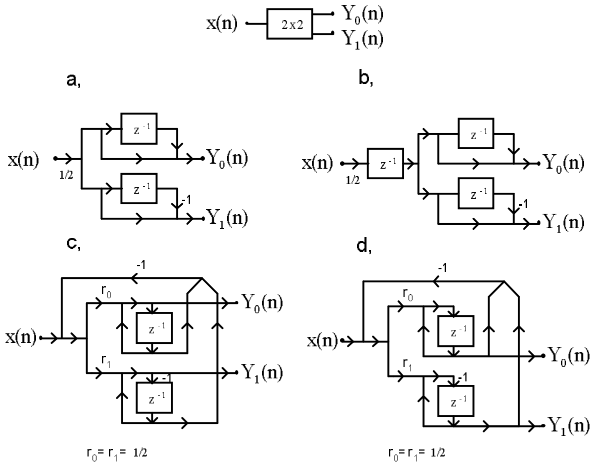

Fig. 2 2*2 DFT filter-bank versions. (a) The simplest tree structure. (b) Tree structure in pipelineable form. (c) Resonator-based version. (d) Resonator-based in pipelineable form.

since the sampling rate ratio at the output stage with respect to the input one is $N/2$, i.e. for the output stage the poles of (8) will be generated. The effects of these poles can be illustrated with the corresponding magnitude and phase characteristics. Fig. 3 presents the magnitude behaviour for different $a>0$ values. Fig. 4 shows the phase characteristics for different a values.

If not only the first stage of the radix-2 structure is replaced with the block of Fig. 2, but all the others, further poles can be introduced. Obviously, depending on the actual stage, the final effect will depend on the sampling rate ratio. If similarly to (9), but with carefully selected ("a") parameters, poles are introduced within every block of every stage, and as a simple extension all the channels get an additional filter of transfer function

$$\frac{1-b^{\frac{2}{N}}}{1-b^{\frac{2}{N}}z^{-2}}, \qquad (10)$$

then a second, completely symmetrical pole-set can be introduced given by

$$\left(1+b^{\frac{1}{2}}z^{-\frac{N}{2}}\right)\left(1+b^{\frac{1}{4}}z^{-\frac{N}{4}}\right)\cdots\left(1+b^{\frac{2}{N}}z^{-2}\right)\left(1-b^{\frac{2}{N}}z^{-2}\right), \qquad (11)$$

which together with (8)

$$\frac{1}{N}\frac{1-a}{1-az^{-N}}\frac{1-b}{1-bz^{-N}}\frac{1-z^{-N}}{1-z^{-1}} \qquad (12)$$

provides an additional parameter for influencing the channel characteristics. Fig. 5 illustrates this effect.

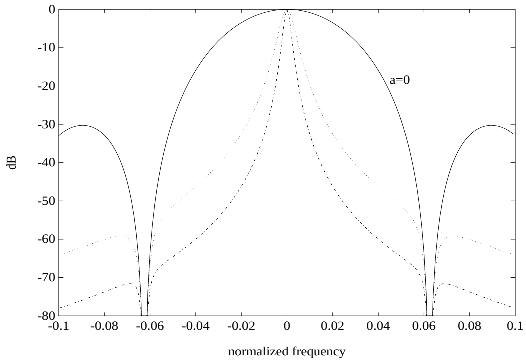

Fig. 3 Magnitude characteristics of the DFT filter-banks with different a>0 parameters.

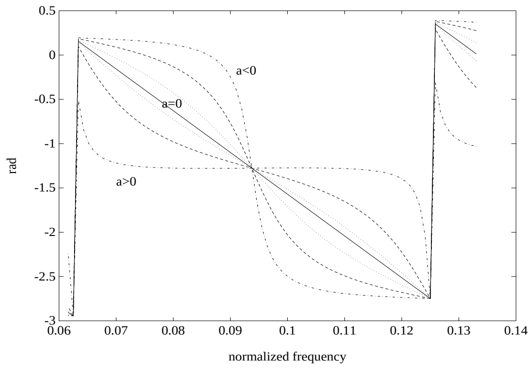

Fig. 4 Phase characteristics of the DFT filter-banks with different a parameters.

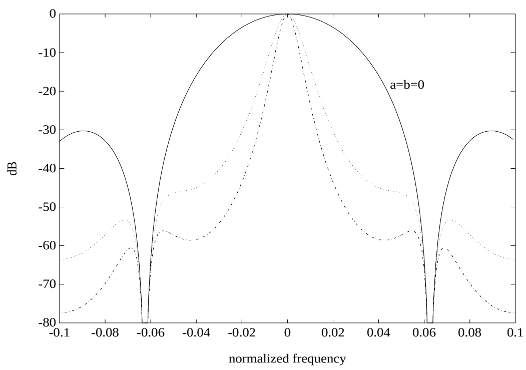

Fig. 5 Magnitude characteristics of the DFT filter-banks with different a, and b parameters.

The proposed DFT structure together with this possibility to introduce poles can serve also as an alternative solution for signal processing problems to be solved in the transform-domain. This means the generalization of the frequency sampling concept (see e.g. [1]) related to the sliding DFT, and the introduction of arbitrary pole-sets by setting the r_0, r_1 parameters properly within the last stage of the overall system. Obviously this latter can be combined with the fading memory effect of (8).

V. CONCLUSIONS

In this paper a new approach to the recursive discrete Fourier transformation has been reported which results in a polyphase structure, and can operate both in sliding window and block recursive modes. The parallelization applied enables decimation and the use of more (in the radix-2 case up to $N/2$) parallel A/D converters, and thus a considerably higher sampling frequency can be achieved with the very same resolution and accuracy. The computational complexity of the new structure corresponds to the fast transformation algorithms. By applying recursive building blocks within the structure the recursive DFT with fading memory can also be implemented. This latter has real importance in such system identification problems where periodic, multi-frequency perturbation signals are applied. In this paper the details are given only for the radix-2 case, however, the ideas are quite general, the extension to other cases is straightforward.

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