Comparison of the Engineers' Fourier transform and the Definition of the Characteristic Function

Some uncertainties in the precise forms of the formulas may stem from the slightly different definitions of the Fourier transform in engineering, and in mathematics. In this appendix, the most important differences are listed.

In the engineering definition, either the variable f or the variable ω is used in the frequency domain. Their relation is $\omega = 2\pi f$. This small difference causes appearance and disappearance of factors $\frac{1}{2\pi}$ if a few formulas.

In the mathematicians' definition of the characteristic function, u is used, which corresponds to ω, but a positive sign is used in the exponent of the kernel of the forward transform. This causes slight changes in a few formulas only.

The differences in the properties of the transform pairs are illustrated in the following expressions.

Definition:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t \qquad \Phi(u) = \int_{-\infty}^{\infty} f(x)e^{jux}\,\mathrm{d}x$$

$$X_\omega(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t \tag{S.1}$$

[0]Printed on February 17, 2006, from the book in preparation: B. Widrow, I. Kollár, ”Quantization Noise.”

Inverse:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, \mathrm{d}f \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u) e^{-jux}\, \mathrm{d}u$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\omega(\omega) e^{j\omega t}\, \mathrm{d}\omega \tag{S.2}$$

Shifting:

$$\begin{aligned}
x(t-a) &\leftrightarrow e^{-j2\pi f a} X(f) \qquad & f(x-a) &\leftrightarrow e^{jua}\, \Phi(u)\\
x(t-a) &\leftrightarrow e^{-j\omega a} X_\omega(\omega) & &\\
e^{j2\pi bt} x(t) &\leftrightarrow X(f-b) & e^{-jbx}\, f(x) &\leftrightarrow \Phi(u-b)\\
e^{j\omega b_\omega t} x(t) &\leftrightarrow X_\omega(\omega - b_\omega) & &
\end{aligned} \tag{S.3}$$

Negative argument (for *real* $x(t)$ and *real* $f(x)$):

$$\begin{aligned}
x(t) &\leftrightarrow \overline{X(-f)} \qquad & f(x) &\leftrightarrow \overline{\Phi(-u)}\\
x(t) &\leftrightarrow \overline{X_\omega(-\omega)} & &\\
x(-t) &\leftrightarrow \overline{X(f)} = X(-f) & f(-x) &\leftrightarrow \overline{\Phi(u)}\\
x(-t) &\leftrightarrow \overline{X_\omega(\omega)} = X_\omega(-\omega) & &
\end{aligned} \tag{S.4}$$

Integral:

$$\int_{-\infty}^{\infty} x(t)\, \mathrm{d}t = X(0) \qquad \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = \Phi(0) = 1$$

$$\int_{-\infty}^{\infty} x(t)\, \mathrm{d}t = X_\omega(0)$$

$$x(0) = \int_{-\infty}^{\infty} X(f)\, \mathrm{d}f \qquad f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u)\, \mathrm{d}u$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\omega(\omega)\, \mathrm{d}\omega \tag{S.5}$$

Derivatives:

$$\begin{aligned}
(-j2\pi t)^n\, x(t) &\leftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}f^n} X(f) \qquad & (jx)^n\, f(x) &\leftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}u^n}\, \Phi(u)\\
(-jt)^n\, x(t) &\leftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} X_\omega(\omega) & &\\
\frac{\mathrm{d}^n}{\mathrm{d}t^n} x(t) &\leftrightarrow (j2\pi f)^n\, X(f) & \frac{\mathrm{d}^n}{\mathrm{d}x^n}\, f(x) &\leftrightarrow (-ju)^n\, \Phi(u)\\
\frac{\mathrm{d}^n}{\mathrm{d}t^n} x(t) &\leftrightarrow (j\omega)^n\, X(\omega) & &
\end{aligned} \tag{S.6}$$

Moments:

$$(-j2\pi)^n \int_{-\infty}^{\infty} t^n x(t)\,\mathrm{d}t \leftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}f^n} X(f)\Big|_{f=0} \qquad (j)^n \int_{-\infty}^{\infty} x^n\, f(x)\,\mathrm{d}x \leftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}u^n}\,\Phi(u)\Big|_{u=0}$$

$$(-j)^n \int_{-\infty}^{\infty} t^n x(t)\,\mathrm{d}t \leftrightarrow \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} X_\omega(\omega)\Big|_{\omega=0} \tag{S.7}$$

Convolution:

$$x_1(t) \star x_2(t) \leftrightarrow X_1(f) X_2(f) \qquad f_1(x) \star f_2(x) \leftrightarrow \Phi_1(u)\,\Phi_2(u)$$

$$x_1(t) \star x_2(t) \leftrightarrow X_{\omega 1}(\omega) X_{\omega 2}(\omega)$$

$$x_1(t) x_2(t) \leftrightarrow X_1(f) \star X_2(f) \qquad f_1(x)\, f_2(x) \leftrightarrow \frac{1}{2\pi}\,\Phi_1(u) \star \Phi_2(u)$$

$$x_1(t) x_2(t) \leftrightarrow \frac{1}{2\pi} X_{\omega 1}(\omega) \star X_{\omega 2}(\omega) \tag{S.8}$$