Solutions for Exercises in Chapter 4

4.1 A program is given in file `problem_4_1.m`.

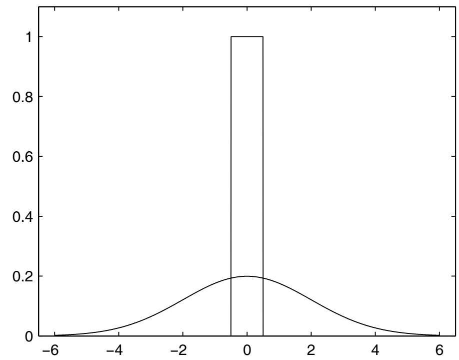

Figure S4.1.1 PDF of the input x with the pulse

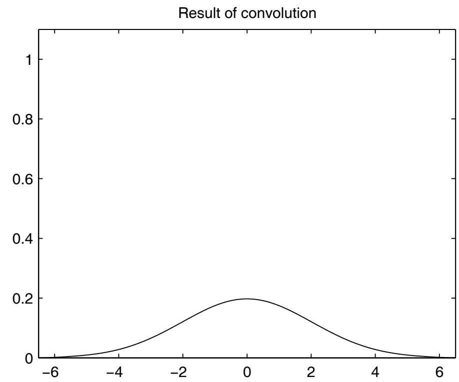

Figure S4.1.2 Result of convolution

4.10 **(a)** The PDF consists of 5 Dirac delta functions at $-2q, -q, 0, q, 2q$, with coefficients 1/32, 8/32, 14/32, 8/32, 1/32, respectively. The characteristic function is

$$\Phi(u) = \frac{14}{32} + \frac{16}{32}\cos(qu) + \frac{2}{32}\cos(2qu) \; . \tag{S4.10.1}$$

(b) The moments of x can be calculated by noticing that the input is a sum of two independent, uniformly distributed random variables in $(-q, q)$. Therefore,

$$\mathrm{E}\{x\} = 0$$
$$\mathrm{E}\{x^2\} = 2\frac{q^2}{3}$$
$$\mathrm{E}\{x^3\} = 0$$

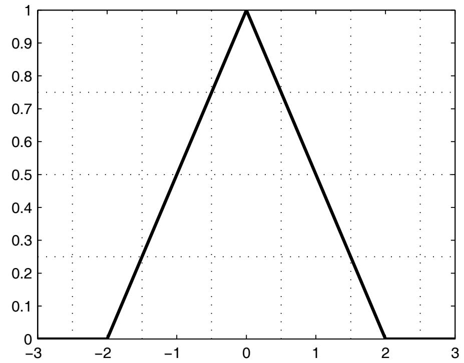

Figure S4.10.1 PDF of input

$$\mathrm{E}\{x^4\} = \frac{q^4}{5} + \frac{q^4}{5} + 6\left(\frac{q^2}{3}\right)^2 = \frac{16}{15}q^4\,. \tag{S4.10.2}$$

The moments of the quantized variable are determined from the discrete PDF:

$$\begin{aligned}
\mathrm{E}\{(x')\} &= 0\\
\mathrm{E}\{(x')^2\} &= 2\frac{1}{32}(2q)^2 + 2\frac{1}{4}q^2 = \frac{3}{4}q^2\\
\mathrm{E}\{(x')^3\} &= 0\\
\mathrm{E}\{(x')^4\} &= 2\frac{1}{32}(2q)^4 + 2\frac{1}{4}q^4 = \frac{3}{2}q^4\,.
\end{aligned} \tag{S4.10.3}$$

(c) Since the input signal fulfils QT III/B ($\Phi(u) = \mathrm{sinc}^2(qu)$), Sheppard's first and second corrections are fulfilled. Indeed,

$$\mathrm{E}\{x\} = \mathrm{E}\{(x')\} - 0, \quad \mathrm{E}\{x^2\} = \mathrm{E}\{(x')^2\} - \frac{1}{12}q^2.$$

Because of the symmetry to zero, the third Sheppard correction is also exactly fulfilled: $\mathrm{E}\{x^3\} = \mathrm{E}\{(x')^3\} - 0$.

Sheppard's fourth correction is not fulfilled. R_4 is not zero:

$$\begin{aligned}
R_4 &= \mathrm{E}\{(x')^4\} - \left(\frac{1}{2}q^2\mathrm{E}\{(x')^2\} - \frac{7}{240}q^4\right) - \mathrm{E}\{x^4\}\\
&= \frac{3}{2}q^4 - \left(\frac{1}{2}q^2\frac{3}{4}q^2 - \frac{7}{240}q^4\right) - \frac{16}{15}q^4\\
&= \frac{21}{240}q^4\\
&= 0.0875\,.
\end{aligned} \tag{S4.10.4}$$

The ratio of the error to the correction is:

$$\frac{R_4}{S_4} = 0.253\,. \tag{S4.10.5}$$

(d) A Monte Carlo experiment is executed in program `problem_4_10.m`.

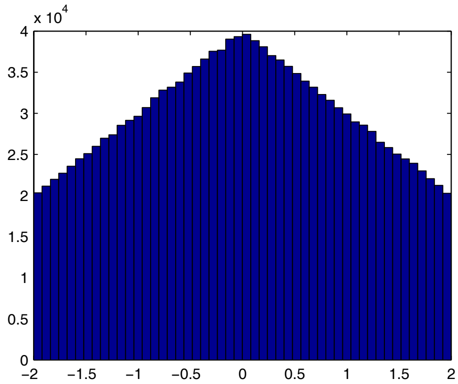

Figure S4.10.2 Histogram of simulated input

4.13 **(a)** The PDF consists of 5 Dirac delta functions at $-2q, -q, 0, q, 2q$, with coefficients

$$\frac{\alpha AB}{32} + \frac{2AB}{8}, \quad \frac{2AB}{4} + \frac{8\alpha AB}{32}, \quad \frac{2AB}{4} + \frac{14\alpha AB}{32}, \quad \frac{2AB}{4} + \frac{8\alpha AB}{32}, \quad \frac{2AB}{8} + \frac{\alpha AB}{32}, \tag{S4.13.1}$$

respectively. The total probability is $2AB + \alpha AB = 1$.

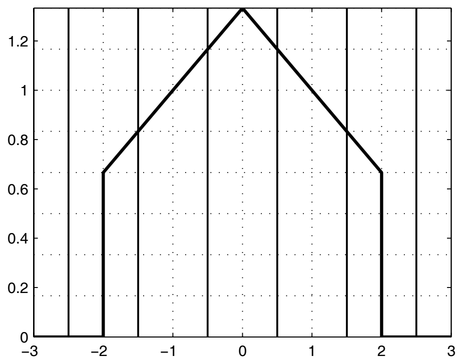

Figure S4.13.1 PDF of input

The characteristic function is

$$\Phi(u) = \left(\frac{2AB}{4} + \frac{14\alpha AB}{32}\right) + 2\left(\frac{2AB}{4} + \frac{8\alpha AB}{32}\right)\cos(qu) + 2\left(\frac{2AB}{8} + \frac{\alpha AB}{32}\right)\cos(2qu)\ . \tag{S4.13.2}$$

(b) The moments of x have been calculated in Exercise 3.12:

$$\begin{aligned}
\mathrm{E}\{x\} &= 0\\
\mathrm{E}\left\{x^2\right\} &= 2AB\frac{A^2}{3} + \alpha AB\frac{A^2}{6}\\
\mathrm{E}\left\{x^3\right\} &= 0\\
\mathrm{E}\left\{x^4\right\} &= 2AB\frac{A^4}{5} + \alpha AB\left(2\frac{A^4}{80} + 6\left(\frac{A^2}{12}\right)^2\right) \qquad \textit{(S3.12.2)}
\end{aligned}$$

The moments of the quantized variable are determined from the discrete PDF:

$$\begin{aligned}
\mathrm{E}\{(x')\} &= 0\\
\mathrm{E}\{(x')^2\} &= 2\left(\frac{2AB}{8} + \frac{\alpha AB}{32}\right)(2q)^2 + 2\left(\frac{2AB}{4} + \frac{8\alpha AB}{32}\right)q^2\\
&= 2AB\frac{3}{2}q^2 + \alpha AB\frac{3}{4}q^2\\
\mathrm{E}\{(x')^3\} &= 0\\
\mathrm{E}\{(x')^4\} &= 2\left(\frac{2AB}{8} + \frac{\alpha AB}{32}\right)(2q)^4 + 2\left(\frac{2AB}{4} + \frac{8\alpha AB}{32}\right)q^4\\
&= 2AB\left(4 + \frac{1}{2}\right)q^4 + \alpha AB\left(1 + \frac{1}{2}\right)q^4\\
&= 2AB\frac{9}{2}q^4 + \alpha AB\frac{3}{2}q^4\,. \qquad \text{(S4.13.3)}
\end{aligned}$$

(c) Since the input signal fulfils QT III/A, Sheppard's first correction is fulfilled. Indeed, $\mathrm{E}\{x\} = \mathrm{E}\{(x')\} - 0$.

The second correction is not valid:

$$\begin{aligned}
R_2 &= \mathrm{E}\{(x')^2\} - S_2 - \mathrm{E}\{x^2\}\\
&= \left(\alpha AB\frac{3}{16}4q^2 + 2AB\frac{3}{8}4q^2\right) - \frac{q^2}{12} - \left(2AB\frac{A^2}{3} + \alpha AB\frac{A^2}{6}\right)\\
&= 2AB\left(\frac{3}{2} - \frac{4}{3}\right)q^2 + \alpha AB\left(\frac{3}{4} - \frac{2}{3}\right)q^2 - \frac{q^2}{12}\\
&= 2AB\frac{1}{6}q^2 + \alpha AB\frac{1}{12}q^2 - \frac{q^2}{12}\\
&= 2AB\frac{q^2}{12}\\
&\approx 0.0556 \qquad \text{(S4.13.4)}
\end{aligned}$$

and this is not zero.

For $\alpha = 1$, $R_2/S_2 = 0.67$.

Because of the symmetry to zero, the third Sheppard correction is exactly fulfilled: $\mathrm{E}\{x^3\} = \mathrm{E}\{(x')^3\} - 0 = 0$.

Sheppard's fourth correction is not fulfilled, either: $R_4 \neq 0$.

$$
\begin{aligned}
R_4 = & \mathrm{E}\{(x')^4\} - \left(\frac{1}{2}q^2\mathrm{E}\{(x')^2\} - \frac{7}{240}q^4\right) - \mathrm{E}\{x^4\} \\
= & 2AB\frac{9}{2}q^4 + \alpha AB\frac{3}{2}q^4 \\
& - \frac{1}{2}q^2\left(2AB\frac{3}{2}q^2 + \alpha AB\frac{3}{4}q^2\right) + \frac{7}{240}q^4 \\
& - \left(2AB\frac{A^4}{5} + \alpha AB\left(2\frac{A^4}{80} + 6\left(\frac{A^2}{12}\right)^2\right)\right) \\
= & 2AB\left(\frac{9}{32} - \frac{3}{64} - \frac{1}{5}\right)A^4 + \alpha AB\left(\frac{3}{32} - \frac{3}{128} - \frac{1}{40} - \frac{1}{24}\right)A^4 + \frac{7}{240}q^4 \\
= & 2AB\frac{132}{3840}A^4 + \alpha AB\frac{14}{3840}A^4 + \frac{7}{240}q^4 \\
\approx & 0.415\,.
\end{aligned}
\tag{S4.13.5}
$$

For $\alpha = 1$, $R_4/S_4 = 0.697$.

(d) A Monte Carlo experiment is executed in program `problem_4_13.m`. A random variable with "house" PDF can be simulated by unifying the set of $N_1 = \frac{2AB}{2AB+\alpha AB}N$ random samples, uniform in $(\pm A)$, with $N_2 = \frac{\alpha AB}{2AB+\alpha AB}N$ random samples, triangular in $(\pm A)$.

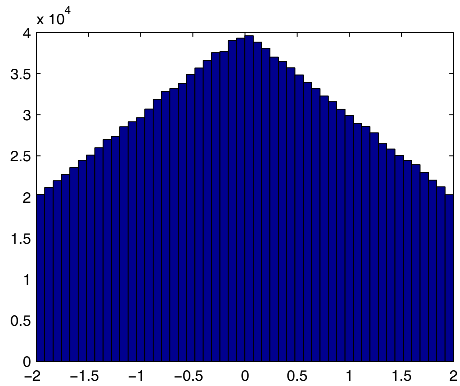

Figure S4.13.2 Histogram of simulated input

4.21 **(a)** We can easily extend the formulas to quantizers with the transfer characteristic shifted along the ideal 45° line. If the size of the shift, as measured on the horizontal axis, is s, the impulse carrier of Eq. (4.5) is slightly modified:

$$
c(x) \triangleq \sum_{m=-\infty}^{\infty} q\delta(x - mq - s)\,. \tag{S4.21.1}
$$

The shift in the exponent causes a phase shift in the CF of the quantized variable with respect to Eq. (4.11):

$$\Phi_{x'}(u) = \Big(\Phi_x(u)\ \mathrm{sinc}\Big(\frac{qu}{2}\Big)\Big) \star \Bigg(\sum_{l=-\infty}^{\infty} e^{jus}\delta(u + l\Psi)\Bigg)$$
$$= \sum_{l=-\infty}^{\infty} e^{-jl\Psi s}\, \Phi_x\,(u + l\Psi)\,\mathrm{sinc}\left(\frac{q(u + l\Psi)}{2}\right), \quad \text{(S4.21.2)}$$

Cf. Eq. (S2.10.2) in the solution of Exercise 2.10. The CF of the quantized variable at the output of a "shifted" quantizer is very similar to the one at the output of a mid-tread quantizer. The central replica is identical, the other repetitions have an additional phase shift. Therefore, all the quantizing theorems hold invariably, independently of s.

(b) For a mid-riser quantizer, for which $s = q/2$, the extra factor is even simpler, $e^{-jl\Psi q/2} = (-1)^l$.

(c) Equation (4.11) can be modified by the exponential terms due to the addition of constant values:

$$\Phi_{x'}(u) = e^{jus} \sum_{l=-\infty}^{\infty} e^{-j(u+l\Psi)s}\, \Phi_x\,(u + l\Psi)\,\mathrm{sinc}\left(\frac{q(u + l\Psi)}{2}\right)$$
$$= \sum_{l=-\infty}^{\infty} e^{-jl\Psi s}\, \Phi_x\,(u + l\Psi)\,\mathrm{sinc}\left(\frac{q(u + l\Psi)}{2}\right). \quad \text{(S4.21.3)}$$

(d) Input offset means that the mean value of the input is apparently increased by μ_{offs}:

$$\Phi_{x'}(u) = \sum_{l=-\infty}^{\infty} e^{j(u+l\Psi)s}\, \Phi_x\,(u + l\Psi)\,\mathrm{sinc}\left(\frac{q(u + l\Psi)}{2}\right). \quad \text{(S4.21.4)}$$

The difference from (S4.21.3) is that this is not corrected for on the quantized side.

(e) This is basically the same problem, except that the sign of the exponent is the opposite. For midrise quantization, this makes no difference.